8 Vectors

8.1 Basic notions

A **vector** is a geometric object that has both magnitude (or length) and direction, in contrast to a **scalar**.

Exercise 48.

Determine whether each of the following quantities in Physics is a vector or a scalar.

- Length (width, height, depth, etc.) vs. displacement vs. distance vs. position
- Time
- Mass
- Temperature
- Amount of substance
- Electric current
- Luminous intensity
- Area vs. (†) vector area
- Volume
- Velocity vs. speed
- Acceleration
- Force
- Moment of force (torque)
- Linear momentum and impulse
- (†) Pressure
- (†) Electric field / magnetic field

In diagrams, a vector is frequently represented by a line segment with a definite direction, or graphically as an arrow, connecting an initial point A with a terminal point B, and denoted by $\overrightarrow{AB}$. The length of the line segment AB is the magnitude of the vector, and the direction from A to B is the direction of the vector.

In this notation, two uppercase letters are used to indicate where the vector starts and ends. Also, vectors are usually denoted in lowercase boldface in prints, such as **a**, **v** and **k**. (Uppercase boldface letters are typically used to represent matrices.) This usage can be seen as a counterpart of using x or y as variables.

Other conventions include $\vec{a}$, $\underline{a}$, $\underset{\sim}{a}$ etc., especially in handwriting.

The vector $\overrightarrow{AB}$ is sometimes also called a **displacement vector** as it can be used to denote the displacement from A to B. Note that $\overrightarrow{AB}$ is not the same as $\overrightarrow{BA}$; in fact, $\overrightarrow{AB} = -\overrightarrow{BA}$. They are in opposite directions.

If a displacement vector starts from the origin O, e.g. $\overrightarrow{OP}$, it is often called a **position vector**, since it identifies the position of its terminal point P. In this case, the corresponding lowercase letter, **p**, is conventionally used as a shorthand notation for the vector $\overrightarrow{OP}$.

Exercise 49.

Draw a regular hexagon $ABCDEF$ with center O.

1. State the relation between the vectors **d** and $\overrightarrow{EF}$.
2. State three vectors that equal to the vector $\overrightarrow{AB}$.

The representations and notations above are of much geometric flavor, while in an algebraic context, a vector (especially a 2-dimensional vector), drawn as an arrow on Cartesian plane, can be "decomposed" into two **components**: instead of moving the vector $\overrightarrow{AB}$ straight along the line AB, one may take a detour by first moving it horizontally to the point C, followed by a vertical move upward to B. In this sense, we may describe the displacement $\overrightarrow{AB}$ as a combination of a horizontal translation and a vertical one. The intrinsic structure of Cartesian plane enables us to further describe such a displacement in a more quantitative way. Suppose the length of the horizontal shift AC is x, and the length of the vertical shift CB is y, then the notation $\begin{pmatrix} x \\ y \end{pmatrix}$ is used to express the vector $\overrightarrow{AB}$. Draw a diagram below to illustrate this idea.

A vector written in the form of $\begin{pmatrix} x \\ y \end{pmatrix}$ is called a **column vector**. This notion can be further generalized to represent vectors in higher dimensions. A 3-dimensional vector represents a displacement in the 3 dimensional space. For example, the vector $\mathbf{p} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$ represents a displacement that decomposes into a one-unit shift in the direction of the x-axis, a two-unit shift in the direction of the y-axis, and a four-unit shift in the opposite direction of the z-axis. When the dimension is higher than 3, vectors are forbiddingly difficult to visualize in a geometric context, but may still embody the similar algebraic structure. One may simply write $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ as a 4-dimensional vector – a vector with 4 components. We can still discuss its length, or more specifically, magnitude. However, its direction becomes visually unclear.

It must be understood that only magnitude and direction are important for a vector – its position is of no significance. This is to say, the vector pointing from $(0, 2)$ to $(-1, 5)$ is the same as the one pointing from $(2, -1)$ to $(1, 2)$, as they are both $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$, when written in the column form.

Exercise 50.

On a Cartesian plane, plot the points: $A(-3, 1)$, $B(-2, 3)$, $C(-2, -2)$, $D(4, 2)$, $G(2, -1)$.

1. Express the vectors $\overrightarrow{AB}$ and $\mathbf{c}$ as column vectors.
2. Given that $\overrightarrow{DE} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$ and that $\overrightarrow{FG} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$, draw these two vectors on the plane, labeling the points E and F.
3. Translate the vector $\overrightarrow{AB}$ such that it starts from the point C, labeling it as $\overrightarrow{CX}$. Now express $\mathbf{x}$ as a column vector.
4. Translate the vector $\mathbf{c}$ such that it starts from the point C, labeling it as $\overrightarrow{CY}$. Now express $\overrightarrow{GY}$ as a column vector.

8.2 Vector algebra

When we view vectors as translations, the sum of two vectors $\mathbf{a}+\mathbf{b}$ could be understood as the overall effect of one translation by $\mathbf{a}$ followed by another translation by $\mathbf{b}$. It is geometrically expressed by drawing the first arrow, representing the vector $\mathbf{a}$, from U to V then drawing the second arrow, representing the vector $\mathbf{b}$, from V to W. Then when the two translations are combined, the sum $\mathbf{a}+\mathbf{b}$ is represented by an arrow from U to W: $\overrightarrow{UV}+\overrightarrow{VW}=\overrightarrow{UW}$.

If we put both vectors in the column form, say $\mathbf{a}=\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{pmatrix}$, it can be seen that

$$\mathbf{a}+\mathbf{b}=\begin{pmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_k+b_k \end{pmatrix}.$$

Draw a diagram below to illustrate **vector addition** in 2 dimensions:

To enlarge or to diminish a translation is basically to change the magnitude of translation while keeping (or sometimes reversing) its direction. This can be done by simply multiplying a vector by a scalar (a number). This is called **scalar multiplication** of vectors.

In the column form, let $\mathbf{a}=\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix}$ be a vector and t be a scalar, then

$$t\mathbf{a}=t\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix}=\begin{pmatrix} ta_1 \\ ta_2 \\ \vdots \\ ta_k \end{pmatrix}.$$

Draw a diagram below to illustrate how a two-dimensional vector $\mathbf{v}$ is multiplied by different scalars: 2, $\frac{1}{2}$ and $-\frac{3}{2}$.

When two vectors $\mathbf{u}$ and $\mathbf{v}$ are in the relation of $\mathbf{u} = k\mathbf{v}$, where k is a non-zero scalar, they are parallel: $\mathbf{u} \parallel \mathbf{v}$, because one of them can be seen as an enlargement or diminishment of the other. The converse is also true: if two vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel, $\mathbf{u} \parallel \mathbf{v}$, then there exists a non-zero scalar k such that $\mathbf{u} = k\mathbf{v}$.

When writing a scalar multiplication $s\mathbf{v}$, conventionally neither a dot nor a cross is put in between, and the scalar s is always put ahead of the vector $\mathbf{v}$. Notations like $s \cdot \mathbf{v}$, $s \times \mathbf{v}$ and $\mathbf{v}s$ are ambiguous and/or misleading, thus are not in use except in some Physics books. Writing $\frac{\mathbf{v}}{s}$ instead of $\frac{1}{s}\mathbf{v}$ is acceptable but not recommended, unless one explicitly defines them to equal.

Vector subtraction is defined in light of vector addition and scalar multiplication:

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-1)\mathbf{b}.$$

In column form, subtraction can be similarly performed componentwise.

$$\begin{pmatrix} k \\ l \end{pmatrix} - \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} k - m \\ l - n \end{pmatrix}.$$

Exercise 51.

1. In the previous exercise, explain how vector addition works for $\overrightarrow{AB} + \overrightarrow{OC}$.
2. In the previous exercise, explain how scalar multiplication works for $2\overrightarrow{OC}$.
3. Draw a rhombus $ABCD$, and denote its center by E. Find a vector on the diagram that represents:
   (a) $-2\overrightarrow{BE}$
   (b) $\mathbf{c} + \overrightarrow{EA}$
   (c) $\mathbf{a} - \mathbf{b}$
   (d) $\overrightarrow{DE} - \overrightarrow{EC}$
   (e) $2\overrightarrow{EC} + \mathbf{a} - \overrightarrow{BC}$

Vector addition and scalar multiplication bear exactly the same properties as those of real numbers:

Property	Numbers	Vectors
Commutativity	$x + y = y + x$	$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
Associativity	$(x + y) + z = x + (y + z)$	$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$
Distributivity	$s(x + y) = sx + sy$	$k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$
	$(x + y)t = xt + yt$	$(k + l)\mathbf{a} = k\mathbf{a} + l\mathbf{a}$

Exercise 52.

Prove these properties by (a) algebraic calculation, and (b) geometrical methods.

8.3 Modulus and unit vectors

The **modulus**, or **magnitude**, of a vector $\mathbf{v}$ represents its length, and is written as $|\mathbf{v}|$.

Given a two-dimensional vector $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, by Pythagorean Theorem, $|\mathbf{v}| = \sqrt{x^2 + y^2}$.

In general, if $\mathbf{v}$ is an n-dimensional vector

$$\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

its modulus is defined to be

$$|\mathbf{v}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Only the zero vector $\mathbf{0}$ has modulus 0. For any non-zero vector $\mathbf{v}$, $|\mathbf{v}| > 0$.

A vector of modulus 1 is called a **unit vector**. A unit vector is often denoted by a lowercase letter with a "hat", such as: $\hat{\mathbf{v}}$. There is a unit vector in every direction.

In the two dimensional Cartesian coordinate system, the unit vectors in the direction of the x- and y-axes are referred to as the 2-dimensional **basic unit vectors**:

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Any 2-dimensional vector can be uniquely written as a linear combination of $\mathbf{i}$ and $\mathbf{j}$: let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ be a 2D vector,

$$\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix} = x\mathbf{i} + y\mathbf{j}.$$

In the three-dimensional space, the basic unit vectors are:

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

Exercise 53.

1. Prove that, for any vectors $\mathbf{a}$ and $\mathbf{b}$,

$$|\mathbf{a}| + |\mathbf{b}| \geq |\mathbf{a} \pm \mathbf{b}| \geq ||\mathbf{a}| - |\mathbf{b}||.$$

   Determine the conditions on which each of the equalities holds.

2. If $\mathbf{p} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{q} = 3\mathbf{i} - 2\mathbf{j}$ are 2-dimensional vectors, calculate the vector $\mathbf{r} = 4\mathbf{p} - \mathbf{q}$ and the unit vector in the direction of $\mathbf{r}$, giving your answers as column vectors.

3. Given the points $A(3, 0, -2)$, $B(5, 1, -1)$ and $C(-1, 2, 0)$, write $\overrightarrow{AB}$, $\frac{1}{2}\overrightarrow{AC}$ and $-\overrightarrow{BC}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. If you start from the point $D(1, 1, -2)$ and the translation $\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC} - \overrightarrow{BC}$ takes you to the point E, find the coordinates of E.

4. Four points A, B, C and D have position vectors $2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$, $3\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$, $-3\mathbf{j} - 2\mathbf{k}$ and $-\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$ respectively. Find the displacement vectors $\overrightarrow{AB}$ and $\overrightarrow{DC}$. What can you deduce about the quadrilateral $ABCD$?

5. If $\mathbf{v} = \begin{pmatrix} x \\ x - 1 \\ 2x \end{pmatrix}$ is a unit vector, find all possible values of x.

6. Prove that, for any angles θ and ϕ, the vector $\begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ \sin\theta \end{pmatrix}$ is a unit vector.

7. Let $\mathbf{u} = x_1\mathbf{i} + y_1\mathbf{j}$ and $\mathbf{v} = x_2\mathbf{i} + y_2\mathbf{j}$. Calculate $|\mathbf{u} - \mathbf{v}|$ and interpret your result geometrically.

8.4 Vector geometry [EXTRA]

Worked examples:

1. Points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$. Find the position vectors of

   (a) the mid-point M of AB,

   (b) the point of trisection T such that $AT = \frac{2}{3}AB$.

   Solution. (a) The displacement vector $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, so $\overrightarrow{AM} = \frac{1}{2}(\mathbf{b} - \mathbf{a})$. Therefore

$$\mathbf{m} = \overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \tfrac{1}{2}\mathbf{a} + \tfrac{1}{2}\mathbf{b}.$$

   (b) The displacement vector $\overrightarrow{AT} = \frac{2}{3}\overrightarrow{AB} = \frac{2}{3}(\mathbf{b} - \mathbf{a})$, so

$$\mathbf{t} = \overrightarrow{OA} + \overrightarrow{AT} = \mathbf{a} + \tfrac{2}{3}(\mathbf{b} - \mathbf{a}) = \tfrac{1}{3}\mathbf{a} + \tfrac{2}{3}\mathbf{b}.$$

□

Remark This method can be modified to find the position vector of an arbitrary point P on the line AB such that $\overrightarrow{AP} = \lambda\overrightarrow{AB}$, where λ is a real number:

$$\mathbf{p} = \overrightarrow{OA} + \overrightarrow{AP} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) = (1 - \lambda)\mathbf{a} + \lambda\mathbf{b}. \qquad (*)$$

Conversely, if the equation $(*)$ holds for some real number λ, then P is a point on the line AB.

2. In triangle ABC the mid-points of BC, CA and AB are D, E and F. Prove that the lines AD, BE and CF (called the **medians**) are **concurrent** – meet at a single point.

   Proof. From the previous example, we know that $\mathbf{d} = \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c}$. Assume now G is the point of intersection of the medians AD and BE. Since it is on AD, its position vector must be in the form of

$$\mathbf{g} = (1 - \lambda)\mathbf{a} + \lambda\mathbf{d} = (1 - \lambda)\mathbf{a} + \tfrac{\lambda}{2}\mathbf{b} + \tfrac{\lambda}{2}\mathbf{c},$$

   where λ is some real number. Similarly, since G is also on BE,

$$\mathbf{g} = (1 - \mu)\mathbf{b} + \mu\mathbf{e} = (1 - \mu)\mathbf{b} + \tfrac{\mu}{2}\mathbf{a} + \tfrac{\mu}{2}\mathbf{c},$$

   where μ is also some real number. Therefore,

$$(1 - \lambda)\mathbf{a} + \tfrac{\lambda}{2}\mathbf{b} + \tfrac{\lambda}{2}\mathbf{c} = (1 - \mu)\mathbf{b} + \tfrac{\mu}{2}\mathbf{a} + \tfrac{\mu}{2}\mathbf{c}.$$

$$\therefore \quad \left(1 - \lambda - \tfrac{\mu}{2}\right)\mathbf{a} + \left(\tfrac{\lambda}{2} - (1 - \mu)\right)\mathbf{b} + \left(\tfrac{\lambda}{2} - \tfrac{\mu}{2}\right)\mathbf{c} = \mathbf{0}. \qquad (*)$$

   Since $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are arbitrary vectors, $(*)$ implies that all coefficients must equal to 0:

$$1 - \lambda - \tfrac{\mu}{2} = \tfrac{\lambda}{2} - (1 - \mu) = \tfrac{\lambda}{2} - \tfrac{\mu}{2} = 0,$$

$$\therefore \quad \lambda = \mu = \tfrac{2}{3},$$

$$\therefore \quad \mathbf{g} = \tfrac{1}{3}\mathbf{a} + \tfrac{1}{3}\mathbf{b} + \tfrac{1}{3}\mathbf{c}.$$

   Now we need to prove that the points C, G and F are collinear.

$$\overrightarrow{CG} = \mathbf{g} - \mathbf{c} = \tfrac{1}{3}\mathbf{a} + \tfrac{1}{3}\mathbf{b} + \tfrac{1}{3}\mathbf{c} - \mathbf{c} = \tfrac{1}{3}\mathbf{a} + \tfrac{1}{3}\mathbf{b} - \tfrac{2}{3}\mathbf{c}, \quad \text{and} \quad \overrightarrow{CF} = \mathbf{f} - \mathbf{c} = \tfrac{1}{2}\mathbf{a} + \tfrac{1}{b}\mathbf{b} - \mathbf{c}.$$

   By comparing the coefficients, we can see that $\overrightarrow{CG} = \frac{2}{3}\overrightarrow{CF}$. This shows that $\overrightarrow{CG} \parallel \overrightarrow{CF}$. Considering that these two vectors share a common point C, we conclude that C, G and F lie on a straight line.

□

Remark The conclusion of this example is that the three medians of a triangle ABC meet at a point G, with position vector $\mathbf{g} = \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$. This point is called the **centroid** of the triangle.

Exercise 54.

1. For each of the following sets of points A, B and C, determine whether the point C lies on the line AB.
   (a) $A(4, 4, 7)$, $B(-2, -5, -5)$, $C(1, 3, 6)$
   (b) $A(4, 1, -1)$, $B(-2, 1, 2)$, $C(6, 1, -2)$

   If yes, draw a diagram showing the relative positions of A, B and C on the line.

2. Four points A, B, C and D have coordinates $(1, 2, -1)$, $(2, 4, 3)$, $(5, 4, 5)$ and $(6, 0, -1)$ respectively. Find the position vectors of
   (a) the mid-point E of AC,
   (b) the point F on BD such that $BF : FD = 1 : 3$.

   Use your answers to draw a sketch showing the relative positions of A, B, C and D.

3. (†) Use vector methods to prove **Menelaus' theorem**:
   Given a triangle ABC, and a transversal line that crosses BC, AC and AB at points D, E and F respectively, with D, E and F distinct from A, B and C, then

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = -1. \qquad (*)$$

   (This equation uses signed lengths of segments, in other words the length AB is taken to be positive or negative according to whether A is to the left or right of B in some fixed orientation of the line. For example, $\frac{AF}{FB}$ is defined as having positive value when F is between A and B and negative otherwise.)
   The converse is also true: if points D, E and F are chosen on BC, AC and AB respectively so that $(*)$ holds, then D, E and F are collinear.

4. (†) Use vector methods to prove **Ceva's theorem**:
   Given a triangle ABC, let the lines AO, BO and CO be drawn from the vertices to a common point O to meet opposite sides at D, E and F respectively. Then, using signed lengths of segments,

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1. \qquad (**)$$

   The converse is also true: If points D, E and F are chosen on BC, AC and AB respectively so that $(**)$ holds, then AD, BE and CF are concurrent.
   (Menelaus' theorem and Ceva's theorem are duals of each other, in the sense that one gives the conditions for three points to fall on a single line, while the other gives the conditions for three lines to intersect in a single point, and these turn out to be the same conditions.)

8.5 Dot product

We first define the **dot product** (also called **scalar product**, or **inner product**) in component form. Generally, for two n-dimensional vectors

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix},$$

their dot product is defined to be

$$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i.$$

Note that the dot product of two vectors is no longer a vector but a scalar.

Remember, whenever you write a dot product, you must put a dot, neither a cross nor nothing, between the vectors.

The algebraic properties of the dot product are counterparts those of multiplying numbers.

Property	Numbers	Vectors
Commutativity	$xy = yx$	$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
Distributivity over addition	$(x + y)z = xz + yz$	$(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$

Exercise 55.

1. Work out the following dot products:

$$\begin{pmatrix} 2 \\ -3 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ -1 \\ 0 \end{pmatrix}, \qquad (2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \cdot (4\mathbf{i} + \mathbf{j} - 2\mathbf{k}).$$

2. Prove the above properties algebraically.

3. Explain whether dot product satisfies associativity.

4. Prove that, when multiplied by scalars, the dot product satisfies

$$(s\mathbf{a}) \cdot (t\mathbf{b}) = (st)(\mathbf{a} \cdot \mathbf{b}).$$

5. Can you deduce, from $\mathbf{u} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w}$, that $\mathbf{u} = \mathbf{v}$?

For a general n-dimensional vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$,

$$\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + \cdots + a_n^2 = |\mathbf{a}|^2. \qquad (*)$$

Or equivalently, $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$.

Now consider two vectors $\mathbf{a}$ and $\mathbf{b}$, starting from a common point and making an angle of θ. The vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c} = \mathbf{a} - \mathbf{b}$ form a triangle. According to the cosine rule, we have

$$|\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos \theta.$$

Substituting dot products for the squared lengths according to $(*)$, we get

$$\begin{aligned} & \mathbf{c} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2|\mathbf{a}||\mathbf{b}| \cos \theta \\ \text{comparing with} \quad & \mathbf{c} \cdot \mathbf{c} = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2(\mathbf{a} \cdot \mathbf{b}) \\ \text{we conclude that} \quad & \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta, \\ \text{or} \quad & \theta = \cos^{-1} \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \right) \end{aligned}$$

Worked examples:

1. Given $\mathbf{a} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, then the angle between $\mathbf{a}$ and $\mathbf{b}$ is

$$\cos^{-1} \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \right) = \cos^{-1} \left(\frac{5 - 10}{\sqrt{26} \cdot \sqrt{29}} \right) \approx 100.5°.$$

2. Given, in three dimensions, $\mathbf{p} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{q} = 4\mathbf{i} + 3\mathbf{k}$, then the angle between $\mathbf{p}$ and $\mathbf{q}$ is

$$\cos^{-1} \left(\frac{\mathbf{p} \cdot \mathbf{q}}{|\mathbf{p}||\mathbf{q}|} \right) = \cos^{-1} \left(\frac{8 + 0 - 6}{\sqrt{9} \cdot \sqrt{25}} \right) \approx 82.3°.$$

3. Given $\mathbf{x} = 2\mathbf{i} + \mathbf{j} - 5\mathbf{k}$ and $\mathbf{y} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$, then $\mathbf{x} \cdot \mathbf{y} = 2 + 3 - 5 = 0$.
Since both $\mathbf{x}$ and $\mathbf{y}$ have positive modulus, the angle θ between them should satisfy $\cos \theta = 0$, therefore $\theta = 90°$. This means $\mathbf{x}$ is perpendicular to $\mathbf{y}$.

4. If $\mathbf{u} = \cos \phi \mathbf{i} + \sin \phi \mathbf{j}$ and $\mathbf{v} = \sin \phi \mathbf{i} - \cos \phi \mathbf{j}$, where ϕ is any angle, then

$$\mathbf{u} \cdot \mathbf{v} = \cos \phi \sin \phi - \sin \phi \cos \phi = 0.$$

Again they are perpendicular vectors.

Remark From the last two examples, we can see that, for two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} \cdot \mathbf{b} = 0$ if and only if $\mathbf{a} \perp \mathbf{b}$. We may apply this argument to prove some geometrical results.

5. In triangle ABC, the **altitudes** AD, BE and CF are concurrent.

Proof. Without loss of generality, let us take the origin O to be the point of intersection of two altitudes AD and BE, and now we need to prove that two lines CO and CF coincide.

$AD \perp BC$ implies $\overrightarrow{OA} \cdot \overrightarrow{BC} = 0$, thus $\mathbf{a} \cdot (\mathbf{c} - \mathbf{b}) = 0$, and $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$.

Similarly $BE \perp AC$ implies $\mathbf{b} \cdot (\mathbf{c} - \mathbf{a}) = 0$, therefore $\mathbf{b} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{a}$.

Combining these two equations yields $\mathbf{a} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{c}$, then $\mathbf{c} \cdot (\mathbf{b} - \mathbf{a}) = 0$ implies $CO \perp AB$, namely CO coincides with the altitude CF.

□

Remark The three altitudes intersect at a single point, called the **orthocenter** of the triangle.

Exercise 56.

1. Which of the following vectors are perpendicular to each other?

$$\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}, \qquad \mathbf{b} = 2\mathbf{i} - 3\mathbf{j} - 6\mathbf{k}, \qquad \mathbf{c} = -3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}, \qquad \mathbf{d} = 6\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}.$$

2. Let $\mathbf{p} = 2\mathbf{i} - 3\mathbf{k}$, $\mathbf{q} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$. Calculate $\mathbf{p} \cdot \mathbf{q}$, $\mathbf{p} \cdot \mathbf{r}$ and $\mathbf{p} \cdot (\mathbf{q} + \mathbf{r})$ and verify that

$$\mathbf{p} \cdot (\mathbf{q} + \mathbf{r}) = \mathbf{p} \cdot \mathbf{q} + \mathbf{p} \cdot \mathbf{r}.$$

3. Calculate the angles between the following pairs of vectors, giving your answers in degrees to one decimal place, where appropriate.

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad \begin{pmatrix} 3 \\ -5 \end{pmatrix} \text{ and } \begin{pmatrix} -5 \\ 3 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ -4 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -4 \\ -5 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} \qquad \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ -3 \end{pmatrix} \text{ and } \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 0 \\ 5 \\ -1 \end{pmatrix}$$

4. The points R, S and T have position vectors $\mathbf{r} = \mathbf{i} - 6\mathbf{j} + \mathbf{k}$, $\mathbf{s} = 3\mathbf{i} - 4\mathbf{k}$, and $\mathbf{t} = \mathbf{i} - 2\mathbf{j}$ respectively. Find the angle RST, thus find the area of the triangle RST.

5. Find the acute angle between the line joining $(3, -2)$ and $(2, 1)$ and the line joining $(1, 4)$ and $(0, 2)$.

6. Find the angle between the diagonals of a cube.

7. In a two-dimensional space, the vector $\mathbf{p}$ is a unit vector, and is perpendicular to the vector $4\mathbf{i} - 3\mathbf{j}$. Find $\mathbf{p}$.

8. The vector $\mathbf{q} = \begin{pmatrix} x \\ x - 1 \\ y \end{pmatrix}$ is perpendicular to both vectors $\begin{pmatrix} 5 \\ -1 \\ y \end{pmatrix}$ and $\begin{pmatrix} 3x \\ x + 1 \\ 0 \end{pmatrix}$.
Find all possible values of x and y.

9. The vectors $\mathbf{a}$ and $\mathbf{b}$ are non-zero. Prove that $(\mathbf{a} + \mathbf{b}) \perp (\mathbf{a} - \mathbf{b})$ if and only if $|\mathbf{a}| = |\mathbf{b}|$.

10. Prove that if a tetrahedron $OABC$ has two pairs of perpendicular opposite edges, the third pair of edges is also perpendicular.

8.6 Cross product [EXTRA]

The **cross product** (also called **vector product**, or **outer product**) of two three-dimensional vectors $\mathbf{a}$ and $\mathbf{b}$ is denoted by $\mathbf{a} \times \mathbf{b}$.

In a three-dimensional Euclidean space, with a right-handed coordinate system, $\mathbf{a} \times \mathbf{b}$ is defined as a vector $\mathbf{c}$ that is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, with a direction given by the **right-handed rule** and a magnitude equal to the area of the parallelogram that the vectors span:

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta\ \hat{\mathbf{n}}, \tag{$*$}$$

where θ is the measure of the angle between $\mathbf{a}$ and $\mathbf{b}$ ($0 \le \theta \le \pi$), and $\hat{\mathbf{n}}$ is the unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, with a direction given by the right-handed rule. The right-handedness constraint is necessary because there exist two unit vectors that are perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, namely, $\hat{\mathbf{n}}$ and $-\hat{\mathbf{n}}$. The cross product $\mathbf{a} \times \mathbf{b}$ is defined so that $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a} \times \mathbf{b}$ becomes a right-handed system (but note that $\mathbf{a}$ and $\mathbf{b}$ are not necessarily orthogonal).

If two vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel, meaning they make an angle of either 0 or π radians, then by definition their cross product of $\mathbf{a}$ and $\mathbf{b}$ is the zero vector $\mathbf{0}$.

Deriving from this definition, the basic unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ satisfy the following equalities:

$$\begin{array}{lll} \mathbf{i} \times \mathbf{i} = \mathbf{0} & \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{i} \times \mathbf{k} = -\mathbf{j} \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{j} \times \mathbf{j} = \mathbf{0} & \mathbf{j} \times \mathbf{k} = \mathbf{i} \\ \mathbf{k} \times \mathbf{i} = \mathbf{j} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} & \mathbf{k} \times \mathbf{k} = \mathbf{0} \end{array}$$

Let two vectors be

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \text{ and } \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

If we assume, for now, distributivity of cross product over vector addition, then their cross product can be calculated as:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &= a_1b_1(\mathbf{i} \times \mathbf{i}) + a_1b_2(\mathbf{i} \times \mathbf{j}) + a_1b_3(\mathbf{i} \times \mathbf{k}) + a_2b_1(\mathbf{j} \times \mathbf{i}) + a_2b_2(\mathbf{j} \times \mathbf{j}) \\ &\quad + a_2b_3(\mathbf{j} \times \mathbf{k}) + a_3b_1(\mathbf{k} \times \mathbf{i}) + a_3b_2(\mathbf{k} \times \mathbf{j}) + a_3b_3(\mathbf{k} \times \mathbf{k}) \\ &= a_1b_1\mathbf{0} + a_1b_2\mathbf{k} + a_1b_3(-\mathbf{j}) + a_2b_1(-\mathbf{k}) + a_2b_2\mathbf{0} + a_2b_3\mathbf{i} + a_3b_1\mathbf{j} + a_3b_2(-\mathbf{i}) + a_3b_3\mathbf{0} \\ &= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}, \end{aligned}$$

or in the column form:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}. \tag{$**$}$$

Both expressions of the cross product ($*$) and ($**$) have to be compatible. The proof is on the next page.

Exercise 57.

1. Find the cross product of $\mathbf{p} = 4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{q} = 3\mathbf{i} - \mathbf{j}$.
2. Find the cross product of $\mathbf{u} = \mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.
3. Find a unit vector that is perpendicular to both $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}$.
4. Find a unit vector that is perpendicular to both $\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 1 \\ -1 \end{pmatrix}$.
5. Prove that for any 3D vectors $\mathbf{a}$ and $\mathbf{b}$,
$$|\mathbf{a} \times \mathbf{b}|^2 + (\mathbf{a} \cdot \mathbf{b})^2 = |\mathbf{a}|^2|\mathbf{b}|^2.$$

(†) Proof of the equivalency of the geometric and algebraic definitions of the cross product:

Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ be two three-dimensional vectors. Define the cross product in the geometric way: $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \hat{\mathbf{n}}$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. Define the operation:

$$\mathbf{a} \otimes \mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}.$$

We need to prove that

$$\mathbf{a} \times \mathbf{b} = \mathbf{a} \otimes \mathbf{b},$$

for all vectors $\mathbf{a}$ and $\mathbf{b}$.

Proof. We first verify that $\mathbf{a} \otimes \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$:

$$\begin{aligned}(\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{a} &= \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \\ &= (a_2b_3 - a_3b_2) \cdot a_1 + (a_3b_1 - a_1b_3) \cdot a_2 + (a_1b_2 - a_2b_1) \cdot a_3 \\ &= a_1a_2b_3 - a_1a_3b_2 + a_2a_3b_1 - a_1a_2b_3 + a_1a_3b_2 - a_2a_3b_1 \\ &= 0.\end{aligned}$$

Similarly one can check that $(\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{b} = 0$. This means:

$$\mathbf{a} \otimes \mathbf{b} \parallel \mathbf{a} \times \mathbf{b}.$$

Now we verify that the modulus of $(\mathbf{a} \otimes \mathbf{b})$ equals that of $(\mathbf{a} \times \mathbf{b})$,

$$\begin{aligned}|\mathbf{a} \otimes \mathbf{b}|^2 &= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \\ &= a_2^2b_3^2 + a_3^2b_2^2 + a_3^2b_1^2 + a_1^2b_3^2 + a_1^2b_2^2 + a_2^2b_1^2 - 2a_2a_3b_2b_3 - 2a_1a_3b_1b_3 - 2a_1a_2b_1b_2. \qquad (*)\end{aligned}$$

On the other hand, we may obtain, from the last exercise question,

$$|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 = (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2. \qquad (**)$$

It is quite straightforward to check that $(**)$ expands to $(*)$, therefore $|\mathbf{a} \otimes \mathbf{b}| = |\mathbf{a} \times \mathbf{b}|$.

Now that $(\mathbf{a} \otimes \mathbf{b})$ and $(\mathbf{a} \times \mathbf{b})$ are parallel and have equal moduli, it remains to show that they are in the same, but not the opposite, direction. This is where right-handedness comes into play.

Since both $(\mathbf{a} \times \mathbf{b})$ and $(\mathbf{a} \otimes \mathbf{b})$ are continuously dependent on the components a_1, a_2, a_3, and b_1, b_2, b_3, it suffices to verify that $\mathbf{i} \times \mathbf{j} = \mathbf{i} \otimes \mathbf{j}$, which is rather obvious.

□

Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be three-dimensional vectors and r be a scalar.

- The cross product is **anticommutative**: $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.

- The cross product is **distributive** over addition: $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$.

- The cross product is **compatible** with scalar multiplication (so that it is **bilinear**): $(r\mathbf{a}) \times \mathbf{b} = r(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (r\mathbf{b})$.

- The cross product is **not associative**: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

  However, the cross product satisfies **Jacobi identity** (see proof on the next page):

  $$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}. \qquad (*)$$

- Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. In particular, for any vector $\mathbf{a}$, $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.

- The cross product does not obey the **cancelation law**. If $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ then, $\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{0}$. It can only be concluded that $\mathbf{a} \parallel (\mathbf{b} - \mathbf{c})$.

  However, if both $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ and $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ hold, then it can be concluded that $\mathbf{b} = \mathbf{c}$. Indeed, we now have both $\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0$ and $\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{0}$, therefore $(\mathbf{b} - \mathbf{c})$ is both parallel and perpendicular to the non-zero vector $\mathbf{a}$. This is only possible if $\mathbf{b} - \mathbf{c} = \mathbf{0}$, namely $\mathbf{b} = \mathbf{c}$.

8.7 (†) Scalar triple product [EXTRA]

The **scalar triple product** is defined as the dot product of one of the vector with the cross product of another two vectors:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

Geometrically, the scalar triple product is the signed volume of the **parallelepiped** defined by the three given vectors.

The scalar triple product has the following property, since each one describes the signed volume of the same parallelepiped.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}).$$

(The parentheses may be omitted without causing ambiguity, since the dot product cannot be evaluated first.)

(†) Proof of Jacobi identity:

Proof. Denote the left hand side of $(*)$ by $\mathbf{d}$. We want to show that $\mathbf{d}$ is perpendicular to all three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, then conclude with the fact that, in the three-dimensional space, a vector perpendicular to three arbitrary vectors must be a zero vector.

$$\begin{aligned}
\mathbf{a} \cdot \mathbf{d} &= \mathbf{a} \cdot (\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b})) \\
&= \mathbf{a} \cdot (\mathbf{a} \times (\mathbf{b} \times \mathbf{c})) + \mathbf{a} \cdot (\mathbf{b} \times (\mathbf{c} \times \mathbf{a})) + \mathbf{a} \cdot (\mathbf{c} \times (\mathbf{a} \times \mathbf{b})) \\
&= (\mathbf{b} \times \mathbf{c}) \cdot (\mathbf{a} \times \mathbf{a}) + (\mathbf{c} \times \mathbf{a}) \cdot (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c}) \qquad (**) \\
&= (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{0} - (\mathbf{a} \times \mathbf{c}) \cdot (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c}) \\
&= 0
\end{aligned}$$

$(**)$ follows the property of the scalar triple product. This means that $\mathbf{d} \perp \mathbf{a}$

Similarly we may prove that $\mathbf{d} \perp \mathbf{b}$ and $\mathbf{d} \perp \mathbf{c}$, thus completing the proof with the argument above.

□

Exercise 58.

1. Given $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$, evaluate the following expressions.

    (a) $\mathbf{a} \times \mathbf{b}$,

    (b) $\mathbf{b} \times (\mathbf{a} - \mathbf{c})$,

    (c) $(\mathbf{b} - 2\mathbf{c}) \times (\mathbf{a} + 3\mathbf{c})$,

    (d) $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$,

    (e) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

2. Points A and B are in the 2-dimensional Cartesian system with coordinates $(1, 2)$ and $(-3, -1)$ respectively. Find the coordinates of point P such that $OAPB$ forms a parallelogram, and evaluate its area.

3. Points A, B and C are in the 3-dimensional Cartesian system, with coordinates $(2, 6, 3)$, $(-2, 0, -5)$ and $(5, 1, -2)$ respectively. Find the coordinates of point D such that $\overrightarrow{AB} \perp \overrightarrow{CD}$ and $\overrightarrow{AD} \times \overrightarrow{BD} = \mathbf{0}$, then evaluate $\overrightarrow{AC} \times \overrightarrow{BD}$.

4. Points A, B and C are in the 3-dimensional Cartesian system, with coordinates $(1, 2, 0)$, $(-3, -1, 0)$ and $(-1, 0, 3)$ respectively. If vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ define a parallelepiped, evaluate its volume.

5. The vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are $\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$ respectively. $\overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{PR}$.

    (a) Find the length of PS.

    (b) Determine the acute angle, in degrees correct to two decimal places, between the diagonals of the parallelogram defined by the points P, Q, R and S.

    (c) Calculate the area of this parallelogram by using the cross product.

6. The points A, B and C have position vectors $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{c} = -2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$ respectively, with respect to a fixed origin. The point D is such that $ABCD$, in that order, is a parallelogram.

    (a) Find the position vector of D.

    (b) Find the position vector of the point E at which the diagonals of the parallelogram intersect.

    (c) Calculate the angle BEC, giving your answer to one decimal place.

    (d) Find the area of the triangle BEC.

    (e) Find the coordinates of any point F such that the line AF is perpendicular to the plane $ABCD$.

7. A three-dimensional unit vector $\mathbf{r}$ has positive components and makes angles of $60°$ with both $\mathbf{i}$ and $\mathbf{j}$.

    (a) Write $\mathbf{r}$ as a column vector.

    (b) State the angle between $\mathbf{r}$ and the unit vector $\mathbf{k}$.

8. The points U, V and W have position vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ respectively relative to the origin O. T is the point on UV such that $\overrightarrow{UV} = 5\overrightarrow{UT}$.

    (a) Show that the position vector of T is $\frac{1}{5}(4\mathbf{u} + \mathbf{v})$.

    (b) Given that the line TW is perpendicular to the line UV, show that

    $$(4\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = 5\mathbf{w} \cdot (\mathbf{u} - \mathbf{v}).$$

    (c) Given also that OU, OV and OW are mutually perpendicular, prove that $2OU = OV$.

9. $ABCD$ is a parallelogram. The coordinates of A, B and D are $(3, 2, 4)$, $(8, 4, 18)$ and $(13, 12, -1)$ respectively. The origin of coordinates is O.

    (a) Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$.

    (b) Find the coordinates of C.

    (c) Show that $\overrightarrow{OA}$ can be expressed in the form $\lambda\overrightarrow{AB} + \mu\overrightarrow{AD}$, stating the values of λ and μ.

    (d) What can you deduce from part (c) about the relative position of the origin and the plane $ABCD$?

8.8 Lines in 2D or 3D

The vector equation of a line in either 2D or 3D, with a given direction $\mathbf{p}$ and passing through a given point $\mathbf{a}$, reads:

$$\mathbf{r} = \mathbf{a} + t\mathbf{p}.$$

Two lines in 3D may be parallel, intersecting or skew.

The angle between two lines is simply the angle between their direction vectors.

One typical question is to find the perpendicular distance from a given point ($\mathbf{u}$) to a given line ($\mathbf{r} = \mathbf{a} + t\mathbf{p}$), for which we have several methods to use:

Method 1 First find the foot of perpendicular:

t_0 is such that $\mathbf{r}_0 = \mathbf{a} + t_0\mathbf{p}$ is the foot of perpendicular. This means $(\mathbf{r}_0 - \mathbf{u}) \perp \mathbf{p}$, namely $(\mathbf{a} + t_0\mathbf{p} - \mathbf{u}) \cdot \mathbf{p}$.

Hence $|\mathbf{r}_0 - \mathbf{u}|$ is the desired distance.

Method 2 Consider the second line that passes through $\mathbf{a}$ and $\mathbf{u}$. Find the angle θ between the two lines involved:

$$\cos\theta = \frac{(\mathbf{u} - \mathbf{a}) \cdot \mathbf{p}}{|\mathbf{u} - \mathbf{a}| \cdot |\mathbf{p}|}.$$

Hence the desired distance may be calculated from $|\mathbf{u} - \mathbf{a}| \sin\theta$.

Method 3 Think from a geometrical point of view by considering the parallelogram spanned by $(\mathbf{u} - \mathbf{a})$ and $\mathbf{p}$, the desired distance d is related to the area of this parallelogram by:

$$d \cdot |\mathbf{p}| = S_{\text{parallelogram}} = |(\mathbf{u} - \mathbf{a}) \times \mathbf{p}|.$$

Hence

$$d = \frac{|(\mathbf{u} - \mathbf{a}) \times \mathbf{p}|}{|\mathbf{p}|}.$$

Exercise 59.

1. Line L_1 passes through the point A and parallels to the vector $\mathbf{p}$, and line L_2 passes through the points B and C. In each of the following cases, determine whether L_1 and L_2:
(i) are the same line; (ii) parallel; (iii) intersect; or (iv) are skew.
   (a) $A(2, -4, 1)$, $B(-5, 2, 0)$, $C(1, -3, 2)$, and $\mathbf{p} = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$
   (b) $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{b} = -2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{c} = 4\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, and $\mathbf{p} = 2\mathbf{i} + \mathbf{j}$
   (c) $A(1, 4, -2)$, $B(2, -3, 3)$, $C(0, 5, -1)$, and $\mathbf{p} = \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}$
   (d) $A(4, 0, 1)$, $B(1, 3, -8)$, $C(6, -2, 7)$, and $\mathbf{p} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$
   (e) $\mathbf{a} = 5\mathbf{i} - \mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $\mathbf{c} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, and $\mathbf{p} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

2. Relative to an origin O, the position vectors of points P and Q are given by
$$\overrightarrow{OP} = \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}, \qquad \text{and} \qquad \overrightarrow{OQ} = \begin{pmatrix} p \\ -3 \\ q \end{pmatrix},$$
where p and q are constants.
   (a) Express the value of $\cos POQ$ in terms of p and q.
   (b) Find the values of p and q for which the length of $\overrightarrow{PQ}$ is 7 units and the angle POQ is a right angle.

   When $p = q = -3$, the line l passes through P and is parallel to OQ.
   (c) State a vector equation for l.
   (d) The point N is the foot of the perpendicular from Q to l, find the position vector of N and evaluate the length of NQ.

3. Relative to an origin O, the position vectors of the points A and B are given by
$$\overrightarrow{OA} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}, \qquad \text{and} \qquad \overrightarrow{OB} = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}.$$
   (a) Use a scalar product to find angle AOB, correct to the nearest degree.
   (b) Find the unit vector in the direction of $\overrightarrow{BA}$.
   (c) The point C is such that $\overrightarrow{OC} = 5\mathbf{i} + p\mathbf{k}$, where p is a constant. Given that the lengths of $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are equal, find the possible values of p.
   (d) The line l has vector equation $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k} + s(\mathbf{i} - 3\mathbf{k})$, determine whether or not the line l intersect the line AB.

4. The points A, B, and C have position vectors, relative to the origin O, given by
$$\overrightarrow{OA} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}, \qquad \overrightarrow{OB} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}, \qquad \overrightarrow{OC} = -4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}.$$
The line l passes through the point C and parallels to the vector $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. The point P lies on l.
   (a) Show that the greatest value of the angle APB is $\frac{\pi}{2}$.
   (b) Find the position vector of P when the angle APB is maximized.
   (c) Determine whether A or B is farther away from the line l.

5. The points L, M, and N have position vectors, relative to the origin O, given by
$$\overrightarrow{OL} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}, \qquad \overrightarrow{OM} = -\mathbf{i} + 2\mathbf{j} + \mathbf{k}, \qquad \overrightarrow{ON} = -2\mathbf{i} - \mathbf{j} + 2\mathbf{k}.$$
Use different methods to find the distance from
   (a) L to the line MN,
   (b) M to the line NL, and
   (c) N to the line LM.