5 Integration

5.1 Integration revisited

If the derivative of the function $F(x)$ is $f(x)$, namely $F'(x) = f(x)$, then $F(x)$ is called **an antiderivative** of $f(x)$.

Any two antiderivatives of $f(x)$ differ only by a constant, thus we introduce the notion of **indefinite integral**, and the following notation:

$$\int f(x)\,\mathrm{d}x = F(x) + C.$$

Each differentiation formula naturally leads to an integration formula:

	$\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}$	$\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + C$, for $n \neq -1$
	$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x) = \mathrm{e}^x$	$\int \mathrm{e}^x\,\mathrm{d}x = \mathrm{e}^x + C$
	$\frac{\mathrm{d}}{\mathrm{d}x}(a^x) = a^x \ln a$, for $a > 0,\ a \neq 1$	$\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a} + C$, for $a > 0,\ a \neq 1$
	$\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x}$	$\int \frac{1}{x}\,\mathrm{d}x = \ln\lvert x\rvert + C$
	$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$	$\int \cos x\,\mathrm{d}x = \sin x + C$
	$\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) =$	$\int \sin x\,\mathrm{d}x =$
	$\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \sec^2 x$	$\int \quad \mathrm{d}x = \tan x + C$
	$\frac{\mathrm{d}}{\mathrm{d}x}(\cot x) =$	$\int \quad \mathrm{d}x =$
	$\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) =$	$\int \quad \mathrm{d}x = \sec x + C$
	$\frac{\mathrm{d}}{\mathrm{d}x}(\csc x) =$	$\int \quad \mathrm{d}x =$
(†)	$\frac{\mathrm{d}}{\mathrm{d}x}(\sin^{-1} x) =$	$\int \quad \mathrm{d}x = \sin^{-1} x + C$
(†)	$\frac{\mathrm{d}}{\mathrm{d}x}(\cos^{-1} x) =$	$\int \quad \mathrm{d}x =$
	$\frac{\mathrm{d}}{\mathrm{d}x}(\tan^{-1} x) =$	$\int \frac{1}{x^2+1}\,\mathrm{d}x =$
	$\int f(x)\,\mathrm{d}x = F(x) + C$	$\int f(ax+b)\,\mathrm{d}x = \frac{F(ax+b)}{a} + C$

Exercise 29.

1. Evaluate the following indefinite integrals.
   (a) $y = \int \frac{2}{3x-1}\,\mathrm{d}x$
   (b) $y = \int \mathrm{e}^{2x+3}\,\mathrm{d}x$
   (c) $y = \int \frac{2}{4x^2+3}\,\mathrm{d}x$
   (d) $y = \int \sin 2x\,\mathrm{d}x$
   (e) $y = \int \cos^2 x\,\mathrm{d}x$
2. The gradient at any point (x, y) on a curve is $4x^3 - 6x + 1$. The curve passes through the point $(-1, 4)$. Find the equation of the curve.

5.2 Definite integrals (with formal treatment) [EXTRA]

The development of the note of the **definite integral** originated from the needs in many practical situations, such as evaluating the area of a given region, evaluating the arc length on a certain curve, and calculating the displacement of an object whose velocity changes over time, etc.

Given the curve with equation $y = x^2$, find the area under the curve, between the lines $x = 0$ and $x = 1$.

(‡) Now let us generalize this idea by investigating a function $f(x)$ on a closed interval $[a, b]$, where $a < b$.

Define a **partition** P as a finite sequence of points in this interval such that,

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$

denote the width of each sub-interval $[x_{k-1}, x_k]$ by $\Delta x_k = x_k - x_{k-1}$, and denote the maximum of these widths by

$$\lambda = \max_{1 \le k \le n} (\Delta x_k).$$

We then take, in each sub-interval, an arbitrary point ξ_k, and sum up the areas of all the small rectangles to obtain the total area:

$$S(P; \xi_k) = \sum_{k=1}^{n} f(\xi_k) \Delta x_k.$$

This $S(P; \xi_k)$ is called a **Riemann sum** of $f(x)$ with respect to the partition P, which is an approximation to the area under the curve over the interval $[a, b]$. It can be seen that such a Riemann sum depends on both the partition P and the points ξ_i chosen in the sub-intervals. However, it is obvious that the actual area is independent of such choices.

When the widths of the sub-intervals become sufficiently small, the Riemann sum is expected to approach the actual area. Hence we define the **definite integral** (or **Riemann integral**) of the function $f(x)$ over the interval $[a, b]$ as the limit of the Riemann sums as the maximum width λ approaches 0:

$$\int_a^b f(x)\, \mathrm{d}x = \lim_{\lambda \to 0} \sum_{k=1}^{n} f(\xi_k) \Delta x_k.$$

(‡) The existence of this limit leads to the issue of **integrability**. This is something way too beyond this course.

Now we return to the area under the curve $y = x^2$ over the interval $[0, 1]$.

To make it simpler, we can use partitions with n sub-intervals of equal length: $\Delta x_k = \frac{1}{n}$, and $x_k = \frac{k}{n}$.

Furthermore, within each sub-interval $[x_{k-1}, x_k]$, the point ξ_k is chosen to be the right endpoint, namely x_k. Then the Riemann sum becomes:

$$
\begin{aligned}
S_n &= \sum_{k=1}^{n} f(\xi_k)\Delta x_k \\
&= \sum_{k=1}^{n} x_k^2 \cdot \frac{1}{n} \\
&= \sum_{k=1}^{n} \frac{1}{n}\left(\frac{k}{n}\right)^2 \\
&= \frac{1}{n^3}\sum_{k=1}^{n} k^2 \\
&= \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} && \text{(result of the summation is used without proof here)} \\
&= \frac{2n^2+3n+1}{6n^2} \\
\Rightarrow \qquad \int_0^1 x^2 \,\mathrm{d}x &= \lim_{n\to\infty} \frac{2n^2+3n+1}{6n^2} && (\lambda = \Delta x_k = \tfrac{1}{n};\ \lambda \to 0 \text{ means } n \to \infty) \\
&= \frac{1}{3}.
\end{aligned}
$$

Exercise 30.

(‡) By applying the same technique, evaluate the integral

$$\int_{-1}^{3} \left(\tfrac{1}{2}x + 1\right) \mathrm{d}x.$$

You may want to verify your answer by finding the corresponding area geometrically.

5.3 Fundamental theorem of calculus [EXTRA]

Taking limits is always something difficult, so we now intuitively introduce the **fundamental theorem of calculus**.

Given that $f(x)$ is a continuous function over the interval $[a, b]$, and that $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(x)\,\mathrm{d}x = F(x)\Big|_a^b = F(b) - F(a).$$

By employing this theorem, we can solve the previous problem in a much quicker way:

$$\int_0^1 x^2\,\mathrm{d}x = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}.$$

Given a function $f(x)$ and real numbers a, b and c, note that

$$\int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x,$$

and that

$$\int_b^a f(x)\,\mathrm{d}x = -\int_a^b f(x)\,\mathrm{d}x.$$

Try to explain these two equations geometrically.

Exercise 31.

Apply the fundamental theorem of calculus to evaluate the following definite integrals.

1. $\displaystyle\int_1^2 \left(x + \frac{1}{x}\right)^3 \mathrm{d}x$
2. $\displaystyle\int_0^1 \mathrm{e}^{2x-1}\,\mathrm{d}x$
3. $\displaystyle\int_0^\pi \sin\frac{x}{2}\,\mathrm{d}x$
4. $\displaystyle\int_0^{\frac{\pi}{4}} \cos^2 3x\,\mathrm{d}x$
5. $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{4x^2+1}\,\mathrm{d}x$
6. $\displaystyle\int_{-1}^1 \frac{1}{x}\,\mathrm{d}x$

(The last question relates to the next section.)

Exercise 32.

1. Find the area of the region enclosed by $y = \sin x$ and $y = \frac{2}{\pi}x$ in the first quadrant.

2. Find the area of the region enclosed by the curve $y = \dfrac{1}{x}$ and the line $2x + 3y = 7$ in the first quadrant.

3. (†) Two functions

$$F(a) = \int_0^a \frac{1}{x^2+1}\,\mathrm{d}x \qquad \text{and} \qquad G(a) = \int_0^a \frac{1}{x+1}\,\mathrm{d}x,$$

are defined for $a > 0$. Find the maximum value of $F(a) - G(a)$.

4. The curve $y = f(x)$ has a stationary point at $(0, 3)$ and it is given that $f''(x) = \mathrm{e}^{\frac{x}{2}}$.

   (a) Find $f(x)$.

   (b) Find the nature of this stationary point.

5. (†) Show that (for $t > 0$)

   (a) $\displaystyle\int_0^1 \frac{1}{(1+tx)^2}\,\mathrm{d}x = \frac{1}{1+t}$,

   (b) $\displaystyle\int_0^1 \frac{-2x}{(1+tx)^3}\,\mathrm{d}x = -\frac{1}{(1+t)^2}$.

   Noting that the right hand side of (b) is the derivative of the right hand side of (a), conjecture the value of

$$\int_0^1 \frac{6x^2}{(1+x)^4}\,\mathrm{d}x.$$

6. (†) Evaluate the integral

$$\int_0^1 (x+1)^{k-1}\,\mathrm{d}x$$

   in the cases $k \neq 0$ and $k = 0$. Hence deduce that $\displaystyle\lim_{k\to 0} \frac{2^k - 1}{k} = \ln 2$.

7. (†) Prove the identities $\cos^4\theta - \sin^4\theta \equiv \cos 2\theta$ and $\cos^4\theta + \sin^4\theta \equiv 1 - \frac{1}{2}\sin^2 2\theta$. Hence evaluate

$$\int_0^{\frac{\pi}{2}} \cos^4\theta\,\mathrm{d}\theta \qquad \text{and} \qquad \int_0^{\frac{\pi}{2}} \sin^4\theta\,\mathrm{d}\theta.$$

   Evaluate also

$$\int_0^{\frac{\pi}{2}} \cos^6\theta\,\mathrm{d}\theta \qquad \text{and} \qquad \int_0^{\frac{\pi}{2}} \sin^6\theta\,\mathrm{d}\theta.$$

8. (†) Find the maximum value of the function

$$f(x) = \int_0^x \frac{1-t}{1+t}\,\mathrm{d}t,$$

   for $x > 0$.

9. (†) Find the derivative of the function

$$f(x) = \int_0^{x^2} \ln(u+1)\,\mathrm{d}u.$$

5.4 Improper integrals

Type I: unbounded function values. For example: $\displaystyle\int_0^1 \frac{\mathrm{d}x}{\sqrt{x}}$.

Type II: unbounded integrating intervals. For example: $\displaystyle\int_1^{+\infty} \frac{\mathrm{d}x}{x^2}$.

Exercise 33.

1. Discuss whether the following improper integrals are finite, in different cases that arise according to the values of the positive constant p.

$$\int_0^1 \frac{1}{x^p}\,\mathrm{d}x; \qquad \int_1^{+\infty} \frac{1}{x^p}\,\mathrm{d}x.$$

2. Evaluate the improper integrals $\displaystyle\int_1^{+\infty} \mathrm{e}^{3-2x}\,\mathrm{d}x$.

3. Determine whether the area under the curve of $y = \tan^2 x$, between $x = 0$ and $x = \frac{1}{2}\pi$, is finite or infinite.

4. (†) It is given that $f(x) = \dfrac{1}{1+x^2}$, for $x \geq 0$.
   (a) Sketch the graph of $f(x)$.
   (b) Find the equation of the line passing through the point $(0, 1)$ and is tangent to the graph of $f(x)$ at another point. Prove that there are no further intersections between the line and the curve.
   (c) By comparing the areas under the graph of $f(x)$ and under this tangent line, prove that $\pi > 3$.
   (d) By comparing the improper integrals

$$\int_1^{+\infty} f(x)\,\mathrm{d}x \qquad \text{and} \qquad \int_1^{+\infty} \frac{1}{x^2}\,\mathrm{d}x,$$

   prove that $\pi < 4$.
   (e) Given that $\tan \frac{1}{3}\pi = \sqrt{3}$ and that $\tan \frac{5}{12}\pi = 2 + \sqrt{3}$, obtain two more (better) upper bounds of π, by comparing similar pairs of improper integrals.
   (f) By expanding $f(x)$ as a power series: $\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots$, and then integrating over the interval $[0, 1]$, show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots = \frac{\pi}{4}.$$

5.5 Volume of revolution

When the region under the graph of $y = f(x)$ between $x = a$ and $x = b$ (where $a < b$) is rotated about the x-axis, the volume of the solid of revolution formed is

$$\int_a^b \pi y^2 \, \mathrm{d}x = \int_a^b \pi \left(f(x)\right)^2 \, \mathrm{d}x.$$

When the region bounded by the (monotonic) curve with equation $y = f(x)$ between the lines $y = c$ and $y = d$ (where $c < d$) is rotated about the y-axis, the volume of the solid of revolution formed is

$$\int_c^d \pi x^2 \, \mathrm{d}y.$$

Such an integral is usually evaluated by first expressing x in terms of y.

(†) One special case is that a decreasing curve cuts the x-axis at $(a, 0)$ and the y-axis at $(0, b)$, where both a and b are positive. When rotating the region enclosed by the curve and the x- and y-axes in the first quadrant, about the y-axis, the volume of revolution can be found by both formulae:

$$\int_0^b \pi x^2 \, \mathrm{d}y, \qquad \text{and} \qquad \int_0^a 2\pi xy \, \mathrm{d}x.$$

This can be proved later by the method of integration by parts.

Exercise 34.

1. Find the volume generated when the region under the graph of $f(x)$ between $x = a$ and $x = b$ is rotated completely about the x-axis, leaving your answers as multiples of π.
   (a) $f(x) = x + \dfrac{1}{\sqrt{x}}$; $a = \frac{1}{4}$, $b = 1$
   (b) $f(x) = \mathrm{e}^{2x-1}$; $a = 0$, $b = 1$
   (c) $f(x) = \cos x$; $a = -\frac{1}{2}\pi$, $b = \frac{1}{2}\pi$
   (d) $f(x) = \tan x$; $a = \frac{1}{4}\pi$, $b = \frac{1}{3}\pi$
2. Find the volume of revolution generated by rotating the region bounded by the curve with equation $y = \mathrm{e}^{-x}$ and the x- and y-axes, through 360° about the x-axis.
3. (†) A **torus** is formed when the interior of a circle with equation $x^2 + (y - r)^2 = a^2$, where r and a are parameters such that $r > a > 0$, is rotated completely about the x-axis. Find the volume of the torus, giving your answers in terms of π, a and r.
4. Find the volume generated when the region bounded by the graph of $f(x)$ between $y = c$ and $y = d$ is rotated completely about the y-axis, leaving your answers as multiples of π.
   (a) $f(x) = \arctan x$; $c = 0$, $d = \frac{\pi}{3}$
   (b) (†) $f(x) = \dfrac{1 - x}{2 + x}$; $c = 0$, $d = \frac{1}{2}$ (Try to use two different methods to solve this question.)

5.6 Integration by partial fractions

Exercise 35.

1. Find the following integrals:

   (a) $\displaystyle\int \frac{x^2 - 4x - 6}{(x-2)(x+3)}\,\mathrm{d}x$

   (b) $\displaystyle\int \frac{x^2 - x}{(x+4)\left(x^2-4\right)}\,\mathrm{d}x$

   (c) $\displaystyle\int \frac{1}{1-x^4}\,\mathrm{d}x$

   (d) $\displaystyle\int_{-\frac{1}{2}}^{1} \frac{1}{(x+1)(x+2)(x+3)}\,\mathrm{d}x$

   (e) (†) $\displaystyle\int_{0}^{+\infty} \frac{x}{(x+1)(x+2)(x+3)}\,\mathrm{d}x$

2. The region enclosed between the graphs of $y = \dfrac{3}{x}$ and $x + 2y = 7$ is denoted by R. Find the volume generated when R is rotated though four right angles about (a) the x-axis; (b) the y-axis.

5.7 Integration by substitution

The first example is rather easy: $\int x(2x+1)^3\,\mathrm{d}x$.

You may evaluate this integral by expanding the cube and integrating term by term. However, if instead we make the substitution: $u = 2x+1$, then $x = \frac{1}{2}(u-1)$. By differentiating x with respect to u, we have $\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{2}$.

Now we write, in terms of **differentials**, $\mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u = \frac{1}{2}\,\mathrm{d}u$, and the integral reads:

$$\begin{aligned}\int x(2x+1)^3\,\mathrm{d}x &= \int \left(\frac{u-1}{2}\cdot u^3\cdot\frac{1}{2}\right)\mathrm{d}u \\ &= \frac{1}{4}\int \left(u^4-u^3\right)\mathrm{d}u \\ &= \frac{1}{4}\left(\frac{u^5}{5}-\frac{u^4}{4}\right)+C \\ &= \frac{(2x+1)^5}{20}-\frac{(2x+1)^4}{16}+C. && \text{(substitute back to } x\text{)}\end{aligned}$$

You may see the benefit of substitution more clearly when you work on this integral: $\int x(2x+1)^{33}\,\mathrm{d}x$.

Exercise 36.

For each of the following substitutions, write $\mathrm{d}x$ in terms of $\phi(t)\,\mathrm{d}t$ or $\mathrm{d}t$ in terms of $\psi(x)\,\mathrm{d}x$, whichever you think is simpler.

1. $t = 1 + x^2$
2. $t = \sqrt{2x+1}$
3. $t = \sin x$
4. $x = \mathrm{e}^t$
5. $x = \ln(t+1)$
6. $t = \tan\frac{x}{2}$
7. $x^2 = \tan t$
8. $\mathrm{e}^t = \cos x$

Worked examples:

1. Find the integral: $\int 3x\sqrt{2x-1}\,\mathrm{d}x$.

   Let $u = \sqrt{2x-1}$, then $x = \dfrac{u^2+1}{2}$, and $\dfrac{\mathrm{d}x}{\mathrm{d}u} = u$, or $\mathrm{d}x = u\,\mathrm{d}u$. Therefore,

$$\begin{aligned}\int 3x\sqrt{2x-1}\,\mathrm{d}x &= \int \left(3\cdot\frac{u^2+1}{2}\cdot u\cdot u\right)\,\mathrm{d}u \\ &= \frac{3}{2}\int \left(u^4+u^2\right)\,\mathrm{d}u \\ &= \frac{3}{2}\left(\frac{u^5}{5}+\frac{u^3}{3}\right)+C \\ &= \frac{3}{10}(2x-1)^{\frac{5}{2}}+\frac{1}{2}(2x-1)^{\frac{3}{2}}+C.\end{aligned}$$

   You may also try to use the substitution $v = 2x-1$, or even without using any substitution at all, to evaluate this integral.

2. Find the integral: $\int \dfrac{1}{x+\sqrt{x}}\,\mathrm{d}x$.

   Let $u = \sqrt{x}$, then $x = u^2$, and $\mathrm{d}x = 2u\,\mathrm{d}u$. Therefore,

$$\begin{aligned}\int \frac{1}{x+\sqrt{x}}\,\mathrm{d}x &= \int \left(\frac{1}{u^2+u}\cdot 2u\right)\,\mathrm{d}u \\ &= \int \frac{2}{u+1}\,\mathrm{d}u \\ &= 2\ln|u+1|+C \\ &= 2\ln\left|\sqrt{x}+1\right|+C.\end{aligned}$$

3. Find the definite integral: $\int_0^1 x^2\sqrt{1-x^3}\,\mathrm{d}x$.

   Let $u = 1-x^3$, then $\mathrm{d}u = -3x^2\,\mathrm{d}x$, or $x^2\,\mathrm{d}x = -\frac{1}{3}\,\mathrm{d}u$.

   When $x = 0$, $u = 1$; when $x = 1$, $u = 0$. It is alright to have the upper limit less than the lower limit. Hence,

$$\begin{aligned}\int_0^1 x^2\sqrt{1-x^3}\,\mathrm{d}x &= \int_1^0 \sqrt{u}\cdot\left(-\frac{1}{3}\right)\,\mathrm{d}u \\ &= \left(-\frac{1}{3}\cdot\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right)\Bigg|_1^0 \\ &= \left(-\frac{2}{9}u^{\frac{3}{2}}\right)\Bigg|_1^0 \\ &= -\frac{2}{9}\cdot(0-1) \\ &= \frac{2}{9}.\end{aligned}$$

   For definite integrals, it is important to substitute also the upper and the lower limits. In this example we see that sometimes it is helpful to write $\mathrm{d}u$ in terms of $\phi(x)\,\mathrm{d}x$, where $\phi(x)$ is a factor of the original integrand.

Exercise 37.

Evaluate the following integrals:

1. $\displaystyle\int \mathrm{e}^{\sin x} \cos x \,\mathrm{d}x$
2. $\displaystyle\int x\sqrt{3-2x^2} \,\mathrm{d}x$
3. $\displaystyle\int \frac{x^2}{x^3-1} \,\mathrm{d}x$
4. $\displaystyle\int_0^1 \frac{x}{x^4+1} \,\mathrm{d}x$
5. $\displaystyle\int_0^{\frac{\pi}{4}} \sin 2x \cos^4 x \,\mathrm{d}x$
6. $\displaystyle\int 3x^3\left(x^2-1\right)^5 \,\mathrm{d}x$
7. $\displaystyle\int \frac{\sqrt{x}+1}{\sqrt{x}-1} \,\mathrm{d}x$
8. $\displaystyle\int_0^1 x^2\sqrt[3]{1-x} \,\mathrm{d}x$
9. $\displaystyle\int_{-1}^0 x^2\sqrt{1-3x} \,\mathrm{d}x$
10. $\displaystyle\int_0^{\frac{\pi}{8}} \sec^4 2x \tan 2x \,\mathrm{d}x$
11. (†) $\displaystyle\int_0^3 \frac{1}{(1+x)\sqrt{x}} \,\mathrm{d}x$
12. (†) $\displaystyle\int_0^1 \frac{1}{\sqrt{x(1-x)}} \,\mathrm{d}x$
13. (†) $\displaystyle\int \frac{1}{\sqrt{x}+\sqrt[3]{x}} \,\mathrm{d}x$
14. (†) $\displaystyle\int \sec x \,\mathrm{d}x$
15. (†) $\displaystyle\int \frac{1}{x^3+1} \,\mathrm{d}x$

5.8 Integration by parts

The formula of **integration by parts** (shortened as IBP) is derived from the product rule of differentiation:

$$(uv)' = u'v + uv'.$$

Integrating both sides with respect to x yields:

$$uv + C = \int (u'v)\,\mathrm{d}x + \int (uv')\,\mathrm{d}x.$$

By rearranging the terms, we can write:

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int \frac{\mathrm{d}u}{\mathrm{d}x}v\,\mathrm{d}x$$

$$\text{or} \qquad \int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u.$$

Note that there are indefinite integrals on both sides, thus it is unnecessary to include the constant '$+C$'.

Worked examples:

1. Find the integral: $\int x\sin 3x\,\mathrm{d}x$.

   The first step is to recognize which part in the integrand is 'u', and which part is '$\mathrm{d}v$', or '$\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x$'.

   Usually when we have a product of a power function (such as x^n) and an exponential, sine, or cosine function (i.e. e^{ax}, $\sin ax$, or $\cos ax$), we consider the power function as u, and the other part (exponential, sine, or cosine together with $\mathrm{d}x$) as $\mathrm{d}v$.

   On the other hand, when a power function x^n is multiplied with a logarithmic function (e.g. $\ln x$), we choose $u = \ln x$, and $\mathrm{d}v = x^n\mathrm{d}x$.

   Therefore in this example, we take $u = x$, and $\mathrm{d}v = \sin 3x\,\mathrm{d}x$, or $\frac{\mathrm{d}v}{\mathrm{d}x} = \sin 3x$. This means that

   $$u = 1, \qquad \text{and} \qquad v = \int \sin 3x\,\mathrm{d}x = -\tfrac{1}{3}\cos 3x\,(+C).$$

   In the following calculation, the above integration constant can be ignored. ((†) Think about why.)

   Now we apply the IBP formula:

   $$\begin{aligned}\int x\sin 3x\,\mathrm{d}x &= \int u\,\mathrm{d}v \\ &= uv - \int v\,\mathrm{d}u \\ &= x\cdot\left(-\tfrac{1}{3}\cos 3x\right) - \int \left(-\tfrac{1}{3}\cos 3x\right)\,\mathrm{d}x \\ &= -\tfrac{1}{3}x\cos 3x + \tfrac{1}{3}\int \cos 3x\,\mathrm{d}x \\ &= -\tfrac{1}{3}x\cos 3x + \tfrac{1}{3}\cdot\tfrac{1}{3}\sin 3x + C \\ &= -\tfrac{1}{3}x\cos 3x + \tfrac{1}{9}\sin 3x + C\end{aligned}$$

2. Find the definite integral: $\int_1^{\mathrm{e}} \ln x\,\mathrm{d}x$.

   First we look at the indefinite integral: $\int \ln x\,\mathrm{d}x$. Set $u = \ln x$, and $\mathrm{d}v = \mathrm{d}x$. This means $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$, and $v = x$. Thus

   $$\int \ln x\,\mathrm{d}x = x\ln x - \int \left(x\cdot\frac{1}{x}\right)\,\mathrm{d}x = x\ln x - \int 1\,\mathrm{d}x = x\ln x - x + C.$$

   Therefore

   $$\int_1^{\mathrm{e}} \ln x\,\mathrm{d}x = (x\ln x - x)\Big|_1^{\mathrm{e}} = (\mathrm{e}-\mathrm{e}) - (0-1) = 1.$$

3. Find the integral: $\displaystyle\int x^2 \mathrm{e}^{-2x}\,\mathrm{d}x$.

Set $u = x^2$, and $\mathrm{d}v = \mathrm{e}^{-2x}\,\mathrm{d}x$, or $\dfrac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{-2x}$. This means

$$\mathrm{d}u = 2x\,\mathrm{d}x, \qquad \text{and} \qquad v = \int \mathrm{e}^{-2x}\,\mathrm{d}x = -\frac{1}{2}\mathrm{e}^{-2x}\;(+C).$$

Now we apply the IBP formula:

$$\begin{aligned}
\int x^2\mathrm{e}^{-2x}\,\mathrm{d}x &= x^2\cdot\left(-\frac{1}{2}\mathrm{e}^{-2x}\right) - \int\left(-\frac{1}{2}\mathrm{e}^{-2x}\right)\cdot 2x\,\mathrm{d}x \\
&= -\frac{x^2}{2}\mathrm{e}^{-2x} + \int x\mathrm{e}^{-2x}\,\mathrm{d}x \\
&= -\frac{x^2}{2}\mathrm{e}^{-2x} + \int x\,\mathrm{d}\left(-\frac{1}{2}\mathrm{e}^{-2x}\right) \\
&= -\frac{x^2}{2}\mathrm{e}^{-2x} + x\cdot\left(-\frac{1}{2}\mathrm{e}^{-2x}\right) - \int\left(-\frac{1}{2}\mathrm{e}^{-2x}\right)\mathrm{d}x \qquad (*) \\
&= -\frac{x^2}{2}\mathrm{e}^{-2x} - \frac{x}{2}\mathrm{e}^{-2x} - \frac{1}{4}\mathrm{e}^{-2x} + C \\
&= -\frac{2x^2+2x+1}{4}\mathrm{e}^{-2x} + C.
\end{aligned}$$

Note that IBP is applied for a second time where the symbol $(*)$ is marked; this time the substitutions of u and v are not explicitly expressed.

4. Find the integral: $\displaystyle\int \mathrm{e}^x \sin x\,\mathrm{d}x$.

For this integral we also need to apply IBP twice:

$$\begin{aligned}
\int \mathrm{e}^x\sin x\,\mathrm{d}x &= \int \sin x\,\mathrm{d}\left(\mathrm{e}^x\right) = \mathrm{e}^x\sin x - \int \mathrm{e}^x\,\mathrm{d}(\sin x) = \mathrm{e}^x\sin x - \int \mathrm{e}^x\cos x\,\mathrm{d}x \\
&= \mathrm{e}^x\sin x - \int \cos x\,\mathrm{d}\left(\mathrm{e}^x\right) = \mathrm{e}^x\sin x - \mathrm{e}^x\cos x + \int \mathrm{e}^x\,\mathrm{d}(\cos x) \\
&= \mathrm{e}^x\sin x - \mathrm{e}^x\cos x - \int \mathrm{e}^x\sin x\,\mathrm{d}x
\end{aligned}$$

$$\begin{aligned}
\therefore \qquad 2\int \mathrm{e}^x\sin x\,\mathrm{d}x &= \mathrm{e}^x\sin x - \mathrm{e}^x\cos x + C' \\
\text{and} \qquad \int \mathrm{e}^x\sin x\,\mathrm{d}x &= \frac{\mathrm{e}^x\sin x - \mathrm{e}^x\cos x}{2} + C.
\end{aligned}$$

One may start with

$$\int \mathrm{e}^x\sin x\,\mathrm{d}x = \int \mathrm{e}^x\,\mathrm{d}(-\cos x)$$

and obtain (obviously) the same answer, still through IBP twice. Try to work out full details by yourself:

Exercise 38.

Evaluate the following integrals:

1. $\displaystyle\int x\mathrm{e}^{3x}\,\mathrm{d}x$
2. $\displaystyle\int x^4 \ln 3x\,\mathrm{d}x$
3. $\displaystyle\int \frac{\ln x}{x^2}\,\mathrm{d}x$
4. $\displaystyle\int_0^{\frac{\pi}{2}} x\sin\frac{x}{2}\,\mathrm{d}x$
5. $\displaystyle\int_0^{\pi} x\sin^2 x\,\mathrm{d}x$
6. $\displaystyle\int x^2\sin 2x\,\mathrm{d}x$
7. $\displaystyle\int \mathrm{e}^{3x}\cos 4x\,\mathrm{d}x$
8. $\displaystyle\int_{-\pi}^{\pi} \mathrm{e}^{-x}\sin 3x\,\mathrm{d}x$
9. $\displaystyle\int_0^1 \tan^{-1} x\,\mathrm{d}x$
10. (†) $\displaystyle\int (x\ln x)^2\,\mathrm{d}x$
11. (†) $\displaystyle\int x^3\mathrm{e}^{x^2}\,\mathrm{d}x$
12. (†) $\displaystyle\int \sin 2x\,\mathrm{e}^{\sin x}\,\mathrm{d}x$
13. (†) $\displaystyle\int_0^1 \frac{\arcsin x}{\sqrt{1-x}}\,\mathrm{d}x$

Exercise 39.

Evaluate each of the following indefinite integrals by substitution.

1. $\displaystyle\int \frac{\mathrm{d}x}{x^2\sqrt{1+x^2}}$. Try different substitutions: (i) $x = \dfrac{1}{t}$; (ii) $x = \tan\theta$; or (iii) first write the integrand as $\dfrac{1}{x^3\sqrt{1+\frac{1}{x^2}}}$, then make the substitution $u = 1 + \dfrac{1}{x^2}$.
2. $\displaystyle\int \tan^{10} x \sec^2 x \,\mathrm{d}x$. Try $u = \tan x$.
3. $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,\mathrm{d}x$
4. $\displaystyle\int \frac{\mathrm{d}x}{\mathrm{e}^x - \mathrm{e}^{-x}}$
5. $\displaystyle\int \frac{\sin x + \cos x}{\sqrt[3]{\sin x - \cos x}}\,\mathrm{d}x$
6. $\displaystyle\int \frac{\mathrm{d}x}{(\arcsin x)^2\sqrt{1-x^2}}$
7. $\displaystyle\int \frac{x\tan\sqrt{1+x^2}}{\sqrt{1+x^2}}\,\mathrm{d}x$
8. $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{1+\mathrm{e}^{2x}}}$. Try $u = \sqrt{1+\mathrm{e}^{2x}}$.
9. $\displaystyle\int \frac{\mathrm{d}x}{x^4\sqrt{1+x^2}}$. Try first substitute $x = \tan\theta$, then $u = \sin\theta$. Alternatively, try $u = \dfrac{x}{\sqrt{1+x^2}}$.
10. $\displaystyle\int \sqrt{\frac{x-1}{x+1}}\,\mathrm{d}x$. First substitute $t = \sqrt{\dfrac{x-1}{x+1}}$, then use partial fraction decomposition.
11. $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{(1-x^2)^3}}$
12. $\displaystyle\int \frac{\mathrm{d}x}{x+\sqrt{1-x^2}}$. First substitute $x = \sin\theta$, then write $\cos\theta$ in the numerator as $\left[\frac{1}{2}(\sin\theta+\cos\theta) + \frac{1}{2}(\cos\theta - \sin\theta)\right]$.
13. $\displaystyle\int \frac{\mathrm{d}x}{1+\sqrt{2x}}$
14. $\displaystyle\int \frac{\mathrm{d}x}{x\sqrt{x^2-1}}$. Try $x = \sec\theta$.
15. $\displaystyle\int \frac{\arctan\sqrt{x}}{\sqrt{x}(1+x)}\,\mathrm{d}x$. Try $u = \arctan\sqrt{x}$.
16. $\displaystyle\int \frac{\mathrm{d}x}{x\sqrt[4]{1+x^4}}$. Try $u^4 = 1 + x^4$.
17. $\displaystyle\int \frac{1+\ln x}{(x\ln x)^2}\,\mathrm{d}x$

Evaluate each of the following indefinite integrals by parts.

18. $\displaystyle\int \mathrm{e}^{\sqrt{x+1}}\,\mathrm{d}x$. First substitute $u = \sqrt{x+1}$.
19. $\displaystyle\int \frac{x}{\sin^2 x}\,\mathrm{d}x$. Hint: $(\cot x)' = -\csc^2 x = -\dfrac{1}{\sin^2 x}$.
20. $\displaystyle\int \arcsin x \,\mathrm{d}x$. First substitute $u = \arcsin x$.
21. $\displaystyle\int x^2 \arctan x \,\mathrm{d}x$

22. $\displaystyle\int \ln^2 x \, \mathrm{d}x$

23. $\displaystyle\int x \tan^2 x \, \mathrm{d}x$. First write $\tan^2 x = \sec^2 x - 1$.

24. $\displaystyle\int \frac{\ln^3 x}{x^2} \, \mathrm{d}x$

25. $\displaystyle\int \cos(\ln x) \, \mathrm{d}x$. First substitute $x = \mathrm{e}^u$.

26. $\displaystyle\int (\arcsin x)^2 \, \mathrm{d}x$

27. $\displaystyle\int \sqrt{x} \mathrm{e}^{\sqrt{x}} \, \mathrm{d}x$

28. $\displaystyle\int \ln(x + \sqrt{1 + x^2}) \, \mathrm{d}x$

(†) Evaluate each of the following indefinite integrals.

29. $\displaystyle\int \frac{x \, \mathrm{d}x}{\sqrt{4x - 3}}$ Try a few alternatives.

30. $\displaystyle\int \frac{\mathrm{d}x}{x(\sqrt[3]{x} - \sqrt{x})}$

31. $\displaystyle\int \frac{\sqrt{1 + x}}{x\sqrt{1 - x}} \, \mathrm{d}x$ Hint: use the substitution $\displaystyle\sqrt{\frac{1 + x}{1 - x}} = t$

32. $\displaystyle\int \frac{\mathrm{d}x}{\sqrt[3]{(x - 1)^2 (x + 1)^4}}$ Hint: write the integrand as $\displaystyle\frac{1}{(x + 1)^2} \cdot \left(\sqrt[3]{\frac{x + 1}{x - 1}} \right)^2$, then substitute $\displaystyle t = \sqrt[3]{\frac{x + 1}{x - 1}}$.

33. $\displaystyle\int \frac{\mathrm{d}x}{4 + 4 \sin x + \cos x}$ Hint: use the substitution $t = \tan \frac{x}{2}$.

34. $\displaystyle\int \ln(1 + x^2) \, \mathrm{d}x$

35. $\displaystyle\int \frac{x + \sin x}{1 + \cos x} \, \mathrm{d}x$ Hint: split into two integrals, then apply IBP for one of them.

36. $\displaystyle\int \frac{\sin^2 x}{\cos^3 x} \, \mathrm{d}x$

37. $\displaystyle\int \frac{\mathrm{d}x}{(1 + \mathrm{e}^x)^2}$

38. (†) Given the following integration formulae:

$$\int \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C; \qquad \int \frac{\mathrm{d}x}{\sqrt{x^2 \pm a^2}} = \ln |x + \sqrt{x^2 \pm a^2}| + C;$$

$$\int \frac{\mathrm{d}x}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right| + C; \qquad \int \frac{\mathrm{d}x}{x^2 + a^2} = \frac{1}{a} \arctan \frac{x}{a} + C;$$

$$\int \sqrt{a^2 - x^2} \, \mathrm{d}x = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a} + C;$$

$$\int \sqrt{x^2 \pm a^2} \, \mathrm{d}x = \frac{1}{2} \left[x\sqrt{x^2 \pm a^2} \pm a^2 \ln \left| x + \sqrt{x^2 \pm a^2} \right| \right] + C;$$

evaluate each of the following indefinite integrals. Hint: write $ax^2 + bc + c$ in the form of $a[(x - p)^2 \pm q^2]$.

$$\int (5x + 3)\sqrt{x^2 + x + 2} \, \mathrm{d}x \qquad \int (x - 1)\sqrt{x^2 + 2x - 5} \, \mathrm{d}x$$

$$\int \frac{(x - 1) \, \mathrm{d}x}{\sqrt{x^2 + x + 1}} \qquad \int \frac{(x + 2) \, \mathrm{d}x}{\sqrt{5 + x - x^2}}$$

Exercise 40.

(†) Enjoy the following challenges.

1. Evaluate the following integrals, in different cases that arise according to the value of the positive constant a:

   (a) $\displaystyle\int_0^1 \frac{1}{x^2+(a+2)x+2a}\,\mathrm{d}x;$

   (b) $\displaystyle\int_1^2 \frac{1}{u^2+au+a-1}\,\mathrm{d}x.$

2. Show, by means of a suitable change of variable, that

$$\int_0^\infty f(\sqrt{x^2+1}+x)\,\mathrm{d}x = \frac{1}{2}\int_1^\infty (1+t^{-2})f(t)\,\mathrm{d}t.$$

   Hence show that

$$\int_0^\infty (\sqrt{x^2+1}+x)^{-3}\,\mathrm{d}x = \frac{3}{8}.$$

3. Let

$$I = \int_0^a \frac{\cos x}{\sin x+\cos x}\,\mathrm{d}x \qquad \text{and} \qquad I = \int_0^a \frac{\sin x}{\sin x+\cos x}\,\mathrm{d}x,$$

   where $0 \le a < \frac{3}{4}\pi$. By considering $I+J$ and $I-J$, show that $2I = a+\ln(\sin a+\cos a)$.
   Find also:

   (a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{p\sin x+q\cos x}\,\mathrm{d}x$, where p and q are positive numbers;

   (b) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x+4}{3\sin x+4\cos x+25}\,\mathrm{d}x.$

4. Differentiate $\sec t$ with respect to t.

   (a) Use the substitution $x=\sec t$ to show that $\displaystyle\int_{\sqrt{2}}^2 \frac{1}{x^3\sqrt{x^2-1}}\,\mathrm{d}x = \frac{\sqrt{3}-2}{8}+\frac{\pi}{24}.$

   (b) Determine $\displaystyle\int \frac{1}{(x+2)\sqrt{(x+1)(x+3)}}\,\mathrm{d}x.$

   (c) Determine $\displaystyle\int \frac{1}{(x+2)\sqrt{x^2+4x-5}}\,\mathrm{d}x.$

5. (a) Use the substitution $u^2=2x+1$ to show that, for $x>4$,

$$\int \frac{3}{(x-4)\sqrt{2x+1}}\,\mathrm{d}x = \ln\left(\frac{\sqrt{2x+1}-3}{\sqrt{2x+1}+3}\right)+K,$$

   where K is a constant.

   (b) Show that $\displaystyle\int_{\ln 3}^{\ln 8} \frac{2}{\mathrm{e}^x\sqrt{\mathrm{e}^x+1}}\,\mathrm{d}x = \frac{7}{12}+\ln\frac{2}{3}.$

6. The variables t and x are related by $t = x+\sqrt{x^2+2bx+c}$, where b and c are constants and $b^2<c$. Show that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{t-x}{t+b},$$

   and hence integrate $\dfrac{1}{\sqrt{x^2+2bx+c}}$.
   Verify by direct integration that your result holds also in the case $b^2=c$ if $x+b>0$ but that your result does not hold in the case $b^2=c$ if $x+b<0$.

7. (a) Show that, for $m>0$,

$$\int_{\frac{1}{m}}^m \frac{x^2}{x+1}\,\mathrm{d}x = \frac{(m-1)^3(m+1)}{2m^2}+\ln m.$$

   (b) Show by means of a substitution that

$$\int_{\frac{1}{m}}^m \frac{1}{x^n(x+1)}\,\mathrm{d}x = \int_{\frac{1}{m}}^m \frac{u^{n-1}}{u+1}\,\mathrm{d}u.$$

(c) Evaluate:

$$\int_{\frac{1}{2}}^{2} \frac{x^5+3}{x^3(x+1)}\,\mathrm{d}x; \qquad \text{and} \qquad \int_{1}^{2} \frac{x^5+x^3+1}{x^3(x+1)}\,\mathrm{d}x.$$

8. Use the substitution $x = \dfrac{1}{t^2-1}$, where $t > 1$, to show that, for $x > 0$,

$$\int \frac{1}{\sqrt{x(x+1)}}\,\mathrm{d}x = 2\ln\left(\sqrt{x}+\sqrt{x+1}\right) + c.$$

[**Note:** You may use without proof the result $\displaystyle\int \frac{1}{t^2-a^2}\,\mathrm{d}t = \frac{1}{2a}\ln\left|\frac{t-a}{t+a}\right| + \text{constant.}$]

The section of the curve $y = \dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt{x+1}}$ between $x = \frac{1}{8}$ and $x = \frac{9}{16}$ is rotated through 360° about the x-axis. Show that the volume enclosed is $2\pi\ln\frac{5}{4}$.

9. The number E is defined by $E = \displaystyle\int_0^1 \frac{\mathrm{e}^x}{1+x}\,\mathrm{d}x$.

Show that $\displaystyle\int_0^1 \frac{x\mathrm{e}^x}{1+x}\,\mathrm{d}x = \mathrm{e} - 1 - E$, and evaluate $\displaystyle\int_0^1 \frac{x^2\mathrm{e}^x}{1+x}\,\mathrm{d}x$ in terms of e and E.

Evaluate also, in terms of E and e as appropriate:

$$\int_0^1 \frac{\mathrm{e}^{\frac{1-x}{1+x}}}{1+x}\,\mathrm{d}x; \qquad \text{and} \qquad \int_1^{\sqrt{2}} \frac{\mathrm{e}^{x^2}}{x}\,\mathrm{d}x.$$

10. The function $f(x)$ is defined by $f(x) = \dfrac{\mathrm{e}^x-1}{\mathrm{e}-1}$, for $x \geq 0$, and the function $g(x)$ is the inverse function to $f(x)$, so that $g(f(x)) = x$. Sketch $f(x)$ and $g(x)$ on the same axes.

Verify, by evaluating each integral, that

$$\int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x + \int_0^k g(x)\,\mathrm{d}x = \frac{1}{2(\sqrt{\mathrm{e}}+1)},$$

where $k = \dfrac{1}{\sqrt{\mathrm{e}}+1}$, and explain this result by means of a diagram.

11. Show that, for any integer m,

$$\int_0^{2\pi} \mathrm{e}^x \cos mx\,\mathrm{d}x = \frac{1}{m^2+1}\left(\mathrm{e}^{2\pi}-1\right).$$

(a) Expand $\cos(A+B) + \cos(A-B)$. Hence show that

$$\int_0^{2\pi} \mathrm{e}^x \cos x \cos 6x\,\mathrm{d}x = \tfrac{19}{650}\left(\mathrm{e}^{2\pi}-1\right).$$

(b) Evaluate $\displaystyle\int_0^{2\pi} \mathrm{e}^x \sin 2x \sin 4x \cos x\,\mathrm{d}x$.

12. Show that

$$\int_0^{\frac{\pi}{4}} \sin(2x)\ln(\cos x)\,\mathrm{d}x = \tfrac{1}{4}(\ln 2 - 1),$$

and that

$$\int_0^{\frac{\pi}{4}} \cos(2x)\ln(\cos x)\,\mathrm{d}x = \tfrac{1}{8}(\pi - \ln 4 - 2).$$

Hence evaluate

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos(2x) + \sin(2x))\ln(\cos x + \sin x)\,\mathrm{d}x.$$

13. (a) Show that, for $n > 0$,

$$\int_0^{\frac{\pi}{4}} \tan^n x \sec^2 x\,\mathrm{d}x = \frac{1}{n+1} \qquad \text{and} \qquad \int_0^{\frac{\pi}{4}} \sec^n x \tan x\,\mathrm{d}x = \frac{(\sqrt{2})^n - 1}{n}.$$

(b) Evaluate the following integrals:

$$\int_0^{\frac{\pi}{4}} x\sec^4 x \tan x\,\mathrm{d}x \qquad \text{and} \qquad \int_0^{\frac{\pi}{4}} x^2 \sec^2 x \tan x\,\mathrm{d}x.$$