1 Algebra

1.1 The modulus function

1.1.1 Definitions and graphs

Define the modulus function and sketch its graph.

Properties:

- $|a| \geq 0$
- $|a||b| = |ab|$, $\dfrac{|a|}{|b|} = \left|\dfrac{a}{b}\right|$
- **Triangle inequalities**: $||a| - |b|| \leq |a \pm b| \leq |a| + |b|$

Exercise 1.

Sketch the following function graphs.

1. $y = |2x + 1|$
2. $y = 2|x + 3| - 1$
3. $y = 2x - |3x + 1|$
4. $y = |x + 1| - |x - 4|$
5. $y = |3x - 2| + |2x + 1|$
6. $y = \left|x^3 + 2x^2 - 2x - 1\right|$
7. $y = \left||x|^3 - 1\right|$
8. $y = \left|\frac{2x-1}{3x+4}\right|$
9. $y = \left|2|x^2 - 1| - 2\right|$
10. $y = \dfrac{|x| - 1}{|x| + 1}$

1.1.2 Equations and inequalities

Exercise 2.

Solve the following equations.

1. $|2x-1|=5$
2. $|3x+1|=2|x-2|$
3. $|x+2|=3x-1$
4. $|2x+1|-x=3$
5. $|x-1|=5-2x$
6. $|x-2|+|2x+1|=\frac{1}{2}x+5$
7. $\left|x^2-3\right|=\dfrac{x+5}{3}$
8. $||x|-2|=1$
9. $x^2+5|x|-6=0$
10. (†) $\quad \left|x^3+x\right|=10$

Exercise 3.

Solve the following inequalities.

1. $|4x-1|>3$
2. $|x+5|\geq 2|x+2|$
3. $|3x-1|-x<7$
4. $|4-x|\leq 2x+1$
5. $|3-x^2|>|x+1|$
6. $|2x-5|\geq 1+|x-1|$
7. $\left|2-x-x^2\right|<1-x$
8. $|3x-2|+|x|\geq 5$
9. $\left|x^2-5x+4\right|>2$
10. (†) $\quad \dfrac{|x|+1}{|x|-2}\leq 3$

1.2 Polynomials

1.2.1 Definitions and operations

Define the following terms:

1. **Polynomial**

2. **Degree**

3. **Coefficient**

Degree of the **zero polynomial** is conventionally defined to be __________________________.

Exercise 4.

1. Determine whether or not each of the followings is a polynomial. Then state, for each polynomial, the degree, the leading term and the constant term.

$$0, \qquad \sqrt{3x+4}, \qquad \frac{x^2-1}{x}, \qquad \left|x(4-x)^2\right|, \qquad 1+x+x^2+x^3+\cdots+x^n+\ldots, \qquad \frac{(x+1)(x+2)\ldots(x+n)}{n!}$$

2. Simplify: $(3x^2+4x+1)(2x^2-x-4)-2(x^3+2x^2-4x-1)$.

3. Given the equation $(x+A)(2x^2-Bx+1)=2x^3+x^2+Cx+2$, find the values of A, B, and C.

4. If $p(x)$ is a polynomial in x of degree m, and $q(x)$ is a polynomial in x of degree n, investigate the degree of

   (a) $p(x)+q(x)$

   (b) $2p(x)-3q(x)$

   (c) $p(x)q(x)$

1.2.2 Division of polynomials

Perform the long division to divide $2x^4 + 3x^2 - x + 2$ by $x^2 - 2x - 1$.

By assuming the coefficients of the quotient and the remainder, divide $3x^5 - x^4 + x^3 + 2x^2 - 4$ by $x^2 + 3x + 2$.

Exercise 5.

Using either method, divide $p(x)$ by $q(x)$.

1. $p(x) = 4x^3 - x - 1$, $q(x) = 2x + 3$
2. $p(x) = 3x^4 + 4x^3 - 2x^2 + 5x - 8$, $q(x) = x^2 - 1$
3. $p(x) = x^6 + x^3 + 1$, $q(x) = x^2 + x + 1$
4. $p(x) = x^n$, $q(x) = x - k$, where n is a positive integer, and k is any real number.

1.2.3 The remainder theorem and the factor theorem

The remainder theorem states that
when a polynomial $p(x)$ is divided by a linear divisor $ax - b$, the remainder is ________.

The factor theorem states that
a linear polynomial $ax - b$ is a factor of the polynomial $p(x)$ **if and only if** ______________.

Exercise 6.

1. Find the remainder when $p(x)$ is divided by $q(x)$.
   (a) $p(x) = 2x^4 - 5x^3 - 2x - 1$, $q(x) = x + 3$
   (b) $p(x) = 3x^3 + 2x^2 - 5x + 1$, $q(x) = 3x - 1$
   (c) $p(x) = 4x^4 - 6x^3 - x^2 + 5x + 1$, $q(x) = 2x - 1$
2. When $x^4 - ax^3 + bx + 1$ is divided by $x - 1$ the remainder is 4. When divided by $x + 2$ the remainder is 3. Find the value of a and b.
3. Given that $x - 2$ is a factor of $p(x) = 6x^3 - 7x^2 + kx - 2$, find the value of k and factorize $p(x)$ completely.
4. Show that $x - 1$ is a factor of $5x^3 + 7x - 12$. Hence solve the inequality $5x^3 + 7x - 12 > 0$.
5. The polynomial $x^5 + 3x^4 + 2x^3 + 2x^2 + 3x - 1$ is denoted by $f(x)$.
   (a) Show that neither $x - 1$ nor $x + 1$ is a factor of $f(x)$.
   (b) Find the quotient and remainder when $f(x)$ is divided by $x^2 - 1$.
   (c) Show that when $f(x)$ is divided by $\left(x^2 + 1\right)$, the remainder is $2x$.
   (d) Hence solve the equation $f(x) = 2x$.

1.3 Binomial series

Recall the binomial theorem:

The general formula for expanding $(x+y)^n$, where n is a positive integer, reads,

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{k}x^{n-k}y^k + \cdots + \binom{n}{n}y^n,$$

where the **binomial coefficients** are given by

$$\binom{n}{k} = \underline{\hspace{6cm}}.$$

Now we are looking at generalizing to the case where n becomes negative or a fraction.

Assuming $\dfrac{1}{1+x} = (1+x)^{-1} = 1 + Ax + Bx^2 + Cx^3 + \ldots$, find the values of A, B and C.

How can you expand $(1+x)^{-1}$ up to and including the term in x^5 then?

Apply the same idea to expand $\sqrt{1+x}$ up to and including the term in x^3.

Simplify the formula for binomial coefficients as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{1\times 2\times\cdots\times k},$$

noting that both the numerator and the denominator are products of k terms.

Calculate the following binomial coefficients for $n=-1$ or $n=\frac{1}{2}$, then compare with your results above.

1. $\binom{n}{1} = \frac{n}{1} =$

2. $\binom{n}{2} = \frac{n(n-1)}{1\times 2} =$

3. $\binom{n}{3} = \frac{n(n-1)(n-2)}{1\times 2\times 3} =$

We conclude, without proof, that for any number n, and $|x|<1$,

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1\times 2}x^2 + \frac{n(n-1)(n-2)}{1\times 2\times 3}x^3 + \cdots + \frac{n(n-1)\cdots(n-k+1)}{1\times 2\times\cdots\times k}x^k + \ldots$$

Think about how to apply this binomial expansion to the following case:

Expand $(4-3x)^{-\frac{1}{2}}$ in ascending powers of x up to and including the term in x^3.

Expand $\left(1-x+2x^2\right)^{\frac{1}{3}}$ in ascending powers of x up to and including the term in x^3.

Exercise 7.

1. Expand the following in ascending powers of x up to and including the term in x^3.

   (a) $\dfrac{1}{(1-3x)^2}$

   (b) $\sqrt[3]{1+2x}$

   (c) $\sqrt{4-x}$

   (d) $(3+2x)^{-3}$

2. Expand the following in ascending powers of x up to and including the term in x^2.

   (a) $(8+3x)^{\frac{2}{3}}$

   (b) $\sqrt[4]{16-x-x^2}$

   (c) $\dfrac{x+1}{2+2x-x^2}$

   (d) (†) $\dfrac{\sqrt{1-3x}}{\sqrt{4+x}}$

3. When expanded in ascending powers of x up to and including the term in x^3, $\dfrac{1}{(1+ax)^3} - \dfrac{1}{(1+2x)^6} = bx^2 + cx^3$. Find the values of a, b, and c.

4. (†) Find the coefficients of x^n in the series expansion of

   (a) $(1+2x)^{-2}$

   (b) $(1-x)^{-\frac{1}{2}}$

   (c) $(1-x)^k$, where k is a negative integer

5. (†) Use the binomial expansion to show that the coefficient of x^r in the expansion of $(1-x)^{-3}$ is $\frac{1}{2}(r+1)(r+2)$.

   (a) Show that the coefficient of x^r in the expansion of $\dfrac{1-x+2x^2}{(1-x)^3}$ is r^2+1 and hence find the sum of the series

   $$1 + \frac{2}{2} + \frac{5}{4} + \frac{10}{8} + \frac{17}{16} + \frac{26}{32} + \frac{37}{64} + \frac{50}{128} + \cdots.$$

   (b) Find the sum of the series

   $$1 + 2 + \frac{9}{4} + 2 + \frac{25}{16} + \frac{9}{8} + \frac{49}{64} + \cdots.$$

1.4 Partial fractions

A **rational function** is a function in the form of $f(x) = \dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are two polynomials in x. **Partial fraction decomposition** is to split a rational function into a sum of simpler fractions.

First make sure the degree of the numerator is less than the degree of the denominator, by applying polynomial division.

When the denominator of the rational function has <u>distinct linear factors</u>, e.g.

$$f(x) = \frac{x^2 + 14x - 18}{(x+2)(x-5)(2x+1)},$$

its partial fraction decomposition is in the form:

$$\frac{A}{x+2} + \frac{B}{x-5} + \frac{C}{2x+1}.$$

Now find the value of the constants:

$$\frac{x^2 + 14x - 18}{(x+2)(x-5)(2x+1)} = \frac{A}{x+2} + \frac{B}{x-5} + \frac{C}{2x+1}$$

$$x^2 + 14x - 18 = A(x-5)(2x+1) + B(x+2)(2x+1) + C(x+2)(x-5) \qquad (*)$$

$$\text{set } x = -2 :$$

$$\text{set } x = 5 :$$

$$\text{set } x = -\frac{1}{2} :$$

An alternative to find the values of the constants is by expanding the right hand side of $(*)$ and then equating the respective coefficients. Try this method by yourself:

Exercise 8.

Express the following functions in partial fractions:

1. $\dfrac{2x+1}{x^3 - x}$
2. $\dfrac{x^2 + 2x^3}{2x^2 - x - 1}$
3. $\dfrac{x^4}{x^2 + x - 2}$
4. $\dfrac{x^2}{(x+1)(x+2)(x+3)}$

When the denominator has repeating linear factors, e.g.

$$f(x) = \frac{3x^3 + 4x^2 + 4x - 3}{(x-1)(x+1)^3},$$

its partial fraction decomposition is in the form:

$$\frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3}.$$

Then find the value of the constants:

Exercise 9.

Express the following functions in partial fractions:

1. $\dfrac{x+1}{x(x-1)^2}$
2. $\dfrac{(x+2)^3}{(x-1)(x-2)^2}$
3. $\dfrac{x^2-7}{x(x+2)^2}$
4. $\dfrac{x^4}{(x+1)^2(x-2)}$
5. $\dfrac{x^2+x+1}{(x^2-1)^2}$

When the denominator has distinct irreducible quadratic factors, e.g.

$$f(x) = \frac{2x^2 - 4x - 5}{(2x+1)(x^2+2x+2)},$$

its partial fraction decomposition is in the form:

$$\frac{A}{2x+1} + \frac{Bx+C}{x^2+2x+2}.$$

Then find the value of the constants:

Exercise 10.

Express the following functions in partial fractions:

1. $\dfrac{x+1}{x(x^2+1)}$

2. $\dfrac{(x+1)^3}{x^3-8}$

3. $\dfrac{x^2}{(x+1)^2(x^2+x+1)}$

4. $\dfrac{2x^4-4x^2+6x-2}{(x-2)(2x^2+2x+1)}$

5. $\dfrac{x^3-11x^2+13x+6}{(x-1)(x^3+8)}$

6. (†) $\dfrac{4x^3-4x}{x^4+4}$

(†) When the denominator of the rational function has repeating irreducible quadratic factors, e.g.

$$f(x) = \frac{x^3+2x-1}{(x+2)(x^2-x+1)^2},$$

conjecture the form of its partial fraction decomposition.

Exercise 11.

1. Express the following functions in partial fractions:

   (a) $y = \dfrac{3x^2}{(x+1)(x+2)(x+4)}$

   (b) $y = \dfrac{x}{(x+1)(x+2)^2}$

   (c) $y = \dfrac{2x^3 - x + 3}{x^2(x-1)}$

   (d) $y = \dfrac{2x^3 - 5x^2 - 1}{(x^2 - x)(x^2 + 1)}$

   (e) $y = \dfrac{x^3 + 3x^2 - 9x - 15}{(x^2 - 9)(x+3)}$

2. By first expressing each of the following expressions in partial fractions, expand in ascending powers of x up to and including the term in x^3:

   (a) $\dfrac{3 - 2x}{(x+2)(x+3)}$

   (b) $\dfrac{x^2 + x + 3}{(x-1)(2x^2 + 3)}$

   (c) $\dfrac{7x^2 + 2x + 5}{(x+1)(3x^2 + 2)}$

3. First simplify then differentiate the expression $\dfrac{x - 15}{x^2 - 25} - \dfrac{5}{x+5} + \dfrac{2}{x-5}$.

4. Split the function $f(x) = \dfrac{x^3 - 5x^2 + 8x + 2}{(x-1)^2(x-4)}$ into partial fractions, then find $f'(x)$.