AMC12/AHSME 2020

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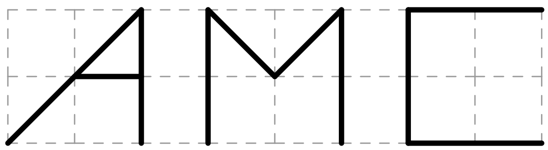
– A

1 Carlos took 70% of a whole pie. Maria took one third of the remainder. What portion of the whole pie was left?

(A) 10% **(B)** 15% **(C)** 20% **(D)** 30% **(E)** 35%

2 The acronym AMC is shown in the rectangular grid below with grid lines spaced 1 unit apart. In units, what is the sum of the lengths of the line segments that form the acronym AMC$?$



(A) 17 **(B)** $15+2\sqrt{2}$ **(C)** $13+4\sqrt{2}$ **(D)** $11+6\sqrt{2}$ **(E)** 21

3 A driver travels for 2 hours at 60 miles per hour, during which her car gets 30 miles per gallon of gasoline. She is paid $\$0.50$ per mile, and her only expense is gasoline at $\$2.00$ per gallon. What is her net rate of pay, in dollars per hour, after this expense?

(A) 20 **(B)** 22 **(C)** 24 **(D)** 25 **(E)** 26

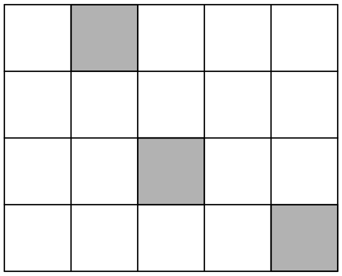
4 How many 4-digit positive integers (that is, integers between 1000 and 9999, inclusive) having only even digits are divisible by $5?$

(A) 80 **(B)** 100 **(C)** 125 **(D)** 200 **(E)** 500

5 The 25 integers from -10 to 14, inclusive, can be arranged to form a 5-by-5 square in which the sum of the numbers in each row, the sum of the numbers in each column, and the sum of the numbers along each of the main diagonals are all the same. What is the value of this common sum?

(A) 2 **(B)** 5 **(C)** 10 **(D)** 25 **(E)** 50

6 In the plane figure shown below, 3 of the unit squares have been shaded. What is the least number of additional unit squares that must be shaded so that the resulting figure has two lines of symmetry?



(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

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7 Seven cubes, whose volumes are $1, 8, 27, 64, 125, 216$, and 343 cubic units, are stacked vertically to form a tower in which the volumes of the cubes decrease from bottom to top. Except for the bottom cube, the bottom face of each cube lies completely on top of the cube below it. What is the total surface area of the tower (including the bottom) in square units?

(A) 644 **(B)** 658 **(C)** 664 **(D)** 720 **(E)** 749

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8 What is the median of the following list of 4040 numbers?

$$1, 2, 3, ..., 2020, 1^2, 2^2, 3^2, ..., 2020^2$$

(A) 1974.5 **(B)** 1975.5 **(C)** 1976.5 **(D)** 1977.5 **(E)** 1978.5

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9 How many solutions does the equation $\tan(2x) = \cos(\frac{x}{2})$ have on the interval $[0, 2\pi]$?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

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10 There is a unique positive integer n such that

$$\log_2(\log_{16} n) = \log_4(\log_4 n).$$

What is the sum of the digits of n?

(A) 4 **(B)** 7 **(C)** 8 **(D)** 11 **(E)** 13

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11 A frog sitting at the point $(1, 2)$ begins a sequence of jumps, where each jump is parallel to one of the coordinate axes and has length 1, and the direction of each jump (up, down, right, or left) is chosen independently at random. The sequence ends when the frog reaches a side of the square with vertices $(0,0)$, $(0,4)$, $(4,4)$, and $(4,0)$. What is the probability that the sequence of jumps ends on a vertical side of the square?

(A) $\frac{1}{2}$ **(B)** $\frac{5}{8}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{3}{4}$ **(E)** $\frac{7}{8}$

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12 Line ℓ in the coordinate plane has the equation $3x - 5y + 40 = 0$. This line is rotated 45° counterclockwise about the point $(20, 20)$ to obtain line k. What is the x-coordinate of the x-intercept of line k?

(A) 10 **(B)** 15 **(C)** 20 **(D)** 25 **(E)** 30

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13 There are integers a, b, and c, each greater than 1, such that

$$\sqrt[a]{N\sqrt[b]{N\sqrt[c]{N}}} = \sqrt[36]{N^{25}}$$

for all $N > 1$. What is b?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

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14 Regular octagon $ABCDEFGH$ has area n. Let m be the area of quadrilateral $ACEG$. What is $\frac{m}{n}$?

(A) $\frac{\sqrt{2}}{4}$ **(B)** $\frac{\sqrt{2}}{2}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{3\sqrt{2}}{5}$ **(E)** $\frac{2\sqrt{2}}{3}$

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15 In the complex plane, let A be the set of solutions to $z^3 - 8 = 0$ and let B be the set of solutions to $z^3 - 8z^2 - 8z + 64 = 0$. What is the greatest distance between a point of A and a point of B?

(A) $2\sqrt{3}$ **(B)** 6 **(C)** 9 **(D)** $2\sqrt{21}$ **(E)** $9 + \sqrt{3}$

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16 A point is chosen at random within the square in the coordinate plane whose vertices are $(0,0)$, $(2020,0)$, $(2020,2020)$, and $(0,2020)$. The probability that the point is within d units of a lattice point is $\frac{1}{2}$. (A point (x,y) is a lattice point if x and y are both integers.) What is d to the nearest tenth?

(A) 0.3 **(B)** 0.4 **(C)** 0.5 **(D)** 0.6 **(E)** 0.7

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17 The vertices of a quadrilateral lie on the graph of $y = \ln x$, and the x-coordinates of these vertices are consecutive positive integers. The area of the quadrilateral is $\ln \frac{91}{90}$. What is the x-coordinate of the leftmost vertex?

(A) 6 **(B)** 7 **(C)** 10 **(D)** 12 **(E)** 13

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18 Quadrilateral $ABCD$ satisfies $\angle ABC = \angle ACD = 90^\circ$, $AC = 20$, and $CD = 30$. Diagonals $\overline{AC}$ and $\overline{BD}$ intersect at point E, and $AE = 5$. What is the area of quadrilateral $ABCD$?

(A) 330 **(B)** 340 **(C)** 350 **(D)** 360 **(E)** 370

19 There exists a unique strictly increasing sequence of nonnegative integers $a_1 < a_2 < \quad < a_k$ such that

$$\frac{2^{289}+1}{2^{17}+1} = 2^{a_1} + 2^{a_2} + \quad + 2^{a_k}.$$

What is k?

(A) 117 **(B)** 136 **(C)** 137 **(D)** 273 **(E)** 306

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20 Let T be the triangle in the coordinate plane with vertices $(0,0)$, $(4,0)$, and $(0,3)$. Consider the following five isometries (rigid transformations) of the plane: rotations of 90°, 180°, and 270° counterclockwise around the origin, reflection across the x-axis, and reflection across the y-axis. How many of the 125 sequences of three of these transformations (not necessarily distinct) will return T to its original position? (For example, a 180° rotation, followed by a reflection across the x-axis, followed by a reflection across the y-axis will return T to its original position, but a 90° rotation, followed by a reflection across the x-axis, followed by another reflection across the x-axis will not return T to its original position.)

(A) 12 **(B)** 15 **(C)** 17 **(D)** 20 **(E)** 25

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21 How many positive integers n are there such that n is a multiple of 5, and the least common multiple of $5!$ and n equals 5 times the greatest common divisor of $10!$ and n?

(A) 12 **(B)** 24 **(C)** 36 **(D)** 48 **(E)** 72

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22 Let (a_n) and (b_n) be the sequences of real numbers such that

$$(2+i)^n = a_n + b_n i$$

for all integers $n \geq 0$, where $i = \sqrt{-1}$. What is

$$\sum_{n=0}^{\infty} \frac{a_n b_n}{7^n}\,?$$

(A) $\frac{3}{8}$ **(B)** $\frac{7}{16}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{9}{16}$ **(E)** $\frac{4}{7}$

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23 Jason rolls three fair standard six-sided dice. Then he looks at the rolls and chooses a subset of the dice (possibly empty, possibly all three dice) to reroll. After rerolling, he wins if and only if the sum of the numbers face up on the three dice is exactly 7. Jason always plays to optimize his chances of winning. What is the probability that he chooses to reroll exactly two of the dice?

(A) $\frac{7}{36}$ **(B)** $\frac{5}{24}$ **(C)** $\frac{2}{9}$ **(D)** $\frac{17}{72}$ **(E)** $\frac{1}{4}$

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24 Suppose that $\triangle ABC$ is an equilateral triangle of side length s, with the property that there is a unique point P inside the triangle such that $AP = 1$, $BP = \sqrt{3}$, and $CP = 2$. What is s?

(A) $1+\sqrt{2}$ **(B)** $\sqrt{7}$ **(C)** $\frac{8}{3}$ **(D)** $\sqrt{5+\sqrt{5}}$ **(E)** $2\sqrt{2}$

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25 The number $a = \frac{p}{q}$, where p and q are relatively prime positive integers, has the property that the sum of all real numbers x satisfying

$$\lfloor x \rfloor \cdot \{x\} = a \cdot x^2$$

is 420, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x. What is $p+q$?

(A) 245 **(B)** 593 **(C)** 929 **(D)** 1331 **(E)** 1332

– B

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1 What is the value in simplest form of the following expression?

$$\sqrt{1} + \sqrt{1+3} + \sqrt{1+3+5} + \sqrt{1+3+5+7}$$

(A) 5 **(B)** $4+\sqrt{7}+\sqrt{10}$ **(C)** 10 **(D)** 15 **(E)** $4+3\sqrt{3}+2\sqrt{5}+\sqrt{7}$

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2 What is the value of the following expression?

$$\frac{100^2-7^2}{70^2-11^2} \cdot \frac{(70-11)(70+11)}{(100-7)(100+7)}$$

(A) 1 **(B)** $\frac{9951}{9950}$ **(C)** $\frac{4780}{4779}$ **(D)** $\frac{108}{107}$ **(E)** $\frac{81}{80}$

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3 The ratio of w to x is $4:3$, the ratio of y to z is $3:2$, and the ratio of z to x is $1:6$. What is the ratio of w to y?

(A) $4:3$ **(B)** $3:2$ **(C)** $8:3$ **(D)** $4:1$ **(E)** $16:3$

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4 The acute angles of a right triangle are a° and b°, where $a > b$ and both a and b are prime numbers. What is the least possible value of b?

(A) 2 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** 11

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5 Teams A and B are playing in a basketball league where each game results in a win for one team and a loss for the other team. Team A has won $\frac{2}{3}$ of its games and team B has won $\frac{5}{8}$ of its games. Also, team B has won 7 more games and lost 7 more games than team A. How many games has team A played?

(A) 21 **(B)** 27 **(C)** 42 **(D)** 48 **(E)** 63

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6 For all integers $n \geq 9,$ the value of

$$\frac{(n+2)! - (n+1)!}{n!}$$

is always which of the following?

(A) a multiple of 4 **(B)** a multiple of 10 **(C)** a prime number
(D) a perfect square **(E)** a perfect cube

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7 Two nonhorizontal, non vertical lines in the xy-coordinate plane intersect to form a 45° angle. One line has slope equal to 6 times the slope of the other line. What is the greatest possible value of the product of the slopes of the two lines?

(A) $\frac{1}{6}$ **(B)** $\frac{2}{3}$ **(C)** $\frac{3}{2}$ **(D)** 3 **(E)** 6

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8 How many ordered pairs of integers (x, y) satisfy the equation

$$x^{2020} + y^2 = 2y?$$

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** infinitely many

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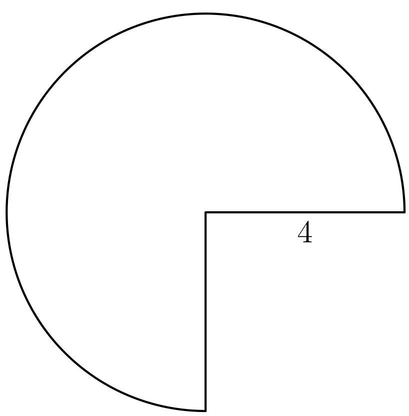
9 A three-quarter sector of a circle of radius 4 inches together with its interior can be rolled up to form the lateral surface area of a right circular cone by taping together along the two radii shown. What is the volume of the cone in cubic inches?



(A) $3\pi\sqrt{5}$ **(B)** $4\pi\sqrt{3}$ **(C)** $3\pi\sqrt{7}$ **(D)** $6\pi\sqrt{3}$ **(E)** $6\pi\sqrt{7}$

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10 In unit square $ABCD$, the inscribed circle ω intersects $\overline{CD}$ at M, and $\overline{AM}$ intersects ω at a point P different from M. What is AP?

(A) $\frac{\sqrt{5}}{12}$ **(B)** $\frac{\sqrt{5}}{10}$ **(C)** $\frac{\sqrt{5}}{9}$ **(D)** $\frac{\sqrt{5}}{8}$ **(E)** $\frac{2\sqrt{5}}{15}$

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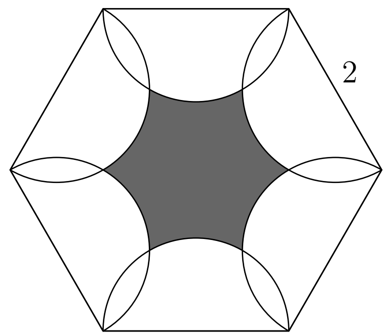
11 As shown in the figure below, six semicircles lie in the interior of a regular hexagon with side length 2 so that the diameters of the semicircles coincide with the sides of the hexagon. What is the area of the shaded regioninside the hexagon but outside all of the semicircles?



(A) $6\sqrt{3}-3\pi$ **(B)** $\frac{9\sqrt{3}}{2}-2\pi$ **(C)** $\frac{3\sqrt{3}}{2}-\frac{\pi}{3}$ **(D)** $3\sqrt{3}-\pi$
(E) $\frac{9\sqrt{3}}{2}-\pi$

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12 Let $\overline{AB}$ be a diameter in a circle of radius $5\sqrt{2}$. Let $\overline{CD}$ be a chord in the circle that intersects $\overline{AB}$ at a point E such that $BE=2\sqrt{5}$ and $\angle AEC=45^\circ$. What is CE^2+DE^2?

(A) 96 **(B)** 98 **(C)** $44\sqrt{5}$ **(D)** $70\sqrt{2}$ **(E)** 100

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13 Which of the following is the value of $\sqrt{\log_2 6+\log_3 6}$?

(A) 1 **(B)** $\sqrt{\log_5 6}$ **(C)** 2 **(D)** $\sqrt{\log_2 3}+\sqrt{\log_3 2}$ **(E)** $\sqrt{\log_2 6}+\sqrt{\log_3 6}$

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14 Bela and Jenn play the following game on the closed interval $[0,n]$ of the real number line, where n is a fixed integer greater than 4. They take turns playing, with Bela going first. At his first turn, Bela chooses any real number in the interval $[0,n]$. Thereafter, the player whose turn it is chooses a real number that is more than one unit away from all numbers previously chosen by either player. A player unable to choose such a number loses. Using optimal strategy, which player will win the game?

(A) Bela will always win. **(B)** Jenn will always win. **(C)** Bela will win if and only if n is odd. **(D)** Jenn will wir
(E) Jenn will win if and only if $n>8$.

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15 There are 10 people standing equally spaced around a circle. Each person knows exactly 3 of the other 9 people: the 2 people standing next to her or him, as well as the person directly across the circle. How many ways are there for the 10 people to split up into 5 pairs so that the members of each pair know each other?

(A) 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 15

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16 An urn contains one red ball and one blue ball. A box of extra red and blue balls lie nearby. George performs the following operation four times: he draws a ball from the urn at random and then takes a ball of the same color from the box and returns those two matching balls to the urn. After the four iterations the urn contains six balls. What is the probability that the urn contains three balls of each color?

(A) $\frac{1}{6}$ **(B)** $\frac{1}{5}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{1}{2}$

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17 How many polynomials of the form $x^5 + ax^4 + bx^3 + cx^2 + dx + 2020$, where a, b, c, and d are real numbers, have the property that whenever r is a root, so is $\frac{-1+i\sqrt{3}}{2} \cdot r$? (Note that $i = \sqrt{-1}$)

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

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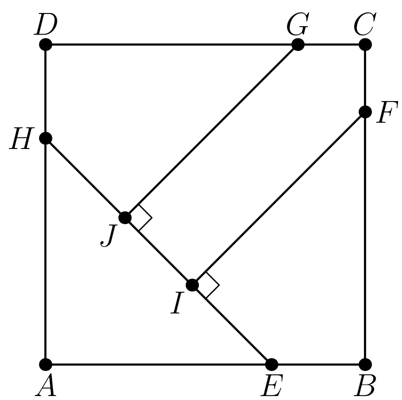
18 In square $ABCD$, points E and H lie on $\overline{AB}$ and $\overline{DA}$, respectively, so that $AE = AH$. Points F and G lie on $\overline{BC}$ and $\overline{CD}$, respectively, and points I and J lie on $\overline{EH}$ so that $\overline{FI} \perp \overline{EH}$ and $\overline{GJ} \perp \overline{EH}$. See the figure below. Triangle AEH, quadrilateral $BFIE$, quadrilateral $DHJG$, and pentagon $FCGJI$ each has area 1. What is FI^2?



(A) $\frac{7}{3}$ **(B)** $8 - 4\sqrt{2}$ **(C)** $1 + \sqrt{2}$ **(D)** $\frac{7}{4}\sqrt{2}$ **(E)** $2\sqrt{2}$

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19 Square $ABCD$ in the coordinate plane has vertices at the points $A(1,1), B(-1,1), C(-1,-1),$ and $D(1,-1)$. Consider the following four transformations:

-L, a rotation of 90° counterclockwise around the origin;
-R, a rotation of 90° clockwise around the origin;
-H, a reflection across the x-axis; and
-V, a reflection across the y-axis.

Each of these transformations maps the squares onto itself, but the positions of the labeled vertices will change. For example, applying R and then V would send the vertex A at $(1,1)$ to $(-1,-1)$ and would send the vertex B at $(-1,1)$ to itself. How many sequences of 20 transformations chosen from $\{L,R,H,V\}$ will send all of the labeled vertices back to their original positions? (For example, R,R,V,H is one sequence of 4 transformations that will send the vertices back to their original positions.)

(A) 2^{37} **(B)** $3\cdot 2^{36}$ **(C)** 2^{38} **(D)** $3\cdot 2^{37}$ **(E)** 2^{39}

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20 Two different cubes of the same size are to be painted, with the color of each face being chosen independently and at random to be either black or white. What is the probability that after they are painted, the cubes can be rotated to be identical in appearance?

(A) $\frac{9}{64}$ **(B)** $\frac{289}{2048}$ **(C)** $\frac{73}{512}$ **(D)** $\frac{147}{1024}$ **(E)** $\frac{589}{4096}$

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21 How many positive integers n satisfy

$$\frac{n+1000}{70}=\lfloor\sqrt{n}\rfloor?$$

(Recall that $\lfloor x\rfloor$ is the greatest integer not exceeding x.)

(A) 2 **(B)** 4 **(C)** 6 **(D)** 30 **(E)** 32

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22 What is the maximum value of $\frac{(2^t-3t)t}{4^t}$ for real values of t?

(A) $\frac{1}{16}$ **(B)** $\frac{1}{15}$ **(C)** $\frac{1}{12}$ **(D)** $\frac{1}{10}$ **(E)** $\frac{1}{9}$

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23 How many integers $n\geq 2$ are there such that whenever $z_1,z_2,...,z_n$ are complex numbers such that

$$|z_1|=|z_2|=...=|z_n|=1 \text{ and } z_1+z_2+...+z_n=0,$$

then the numbers $z_1,z_2,...,z_n$ are equally spaced on the unit circle in the complex plane?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

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24 Let $D(n)$ denote the number of ways of writing the positive integer n as a product

$$n=f_1\cdot f_2\cdots f_k,$$

where $k\geq 1$, the f_i are integers strictly greater than 1, and the order in which the factors are listed matters (that is, two representations that differ only in the order of the factors are counted as distinct). For example, the number 6 can be written as 6, $2\cdot 3$, and $3\cdot 2$, so $D(6)=3$. What is $D(96)$?

(A) 112 **(B)** 128 **(C)** 144 **(D)** 172 **(E)** 184

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25 For each real number a with $0 \le a \le 1$, let numbers x and y be chosen independently at random from the intervals $[0, a]$ and $[0, 1]$, respectively, and let $P(a)$ be the probability that

$$\sin^2(\pi x) + \sin^2(\pi y) > 1.$$

What is the maximum value of $P(a)$?

(A) $\frac{7}{12}$ **(B)** $2 - \sqrt{2}$ **(C)** $\frac{1+\sqrt{2}}{4}$ **(D)** $\frac{\sqrt{5}-1}{2}$ **(E)** $\frac{5}{8}$
